# Fourier transform of subanalytic functions

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# General motivation: logic & geometry vs integration

Study

$$\mathbb{R}^m \supset X \ni x \mapsto \mathscr{I}(x) = \int_{y \in \mathbb{R}^n} f(x, y) \, \mathrm{d}y,$$

when $f$ belongs to a certain class of tame functions, to understand how the integration process destroys or preserves the geometric properties of $f$.

Strategy. Prepare the function $f$ on cells and deal with the asymptotics.

A possible framework: a class of functions built on subanalytic functions to use the Parusiński/Lion-Rolin preparation theorem.

## Specific motivation: oscillatory integrals

$$\boxed{\mathbb{R}^m \supset x \mapsto \mathscr{I}(x) = \int_{y\in\mathbb{R}^n} e^{\mathrm{i}\varphi(x,y)} f(x,y)\,\mathrm{d}y.}$$

Example 1. $m = 1$ and $\mathscr{I}(x) = \int_{y\in\mathbb{R}^n} e^{\mathrm{i}x\phi(y)} f(y)\,\mathrm{d}y$.

- *Phase* $\phi$ : analytic, $\phi(0) = 0$, $0 \in \mathbb{R}^n$ an isolated singularity $\phi$,
- *Amplitude* $f$ : $\mathscr{C}^\infty$ with compact support.

In singularity theory. (Arnol'd-Gusein-Zade-Varchenko, Malgrange, E. M. Stein...) Asymptotics of $\mathscr{I}(x)$ when $x \to +\infty$ vs singularity of $\phi$. After resolution of singularities of $\phi$:

$$\exists r \in \mathbb{N}^*, \qquad \mathscr{I}(x) \underset{x\to+\infty}{\sim} \sum_{p\in\mathbb{N}^*} x^{-p/r} \sum_{k=0}^{n-1} a_{p,k} \log^k x.$$

$a_{p,k} \neq 0 \Longleftrightarrow e^{2i\pi(\frac{p}{r}-1)}$ eigenvalue of the monodromy of $\phi$ of mult. $\geq k+1$.
Example 2. Parametric Fourier transform

$$\widehat{f}(x) = \int_{y\in\mathbb{R}^n} e^{-2\pi\mathrm{i}(x\cdot y)} f(x,y)\,\mathrm{d}y.$$

# Subanalytic sets and functions

Subanalytic sets. $\mathscr{S} = (\mathscr{S}_n)_{n\in\mathbb{N}}$ minimal s.t.

- $\mathbb{R}^n \in \mathscr{S}_n \subset \mathscr{P}(\mathbb{R}^n)$,
- $\mathscr{S}_n$ contains all algebraic sets and is stable under finite union and complement,
- $\mathscr{S}$ stable under projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and product,
- $\mathscr{S}_{n+1}$ contains all graphs of restricted analytic functions $f : [-1, 1]^n \to \mathbb{R}$.

Examples. $[1, +\infty[ \ni x \mapsto \log(1 + \frac{1}{x}) \in \mathscr{S}_2$, $y = \sin x$, $y = \log x \notin \mathscr{S}_2$.

Subanalytic functions. Functions with s.a. graph.

$$X \text{ s.a.} \subset \mathbb{R}^n, \quad \mathscr{S}(X) := \{f : X \to \mathbb{R};\ f \text{ s.a.}\}.$$

Remark. $f$ s.a. $\implies \exists p \in \mathbb{Q}$, $a \in \mathbb{R}$, $f(x) \sim_\infty ax^p$.

Theorem (Parusiński, Lion-Rolin).

- *s.a. sets split in a finite number of analytic cells,*
- *s.a. functions can be prepared on such cells:*

$$f(x, y) = (y - \theta(x))^q h(x) u(x, y).$$

# What is already known?

Basic question ($\varphi = 0$): Class of $\mathscr{I}$ when $f \in \mathscr{S}(X \times \mathbb{R}^n)$?

Notation.

- For $X \subseteq \mathbb{R}^m$ and $f : X \times \mathbb{R}^n \to \mathbb{R}$ denote

$$\mathrm{Int}(f) := \{x \in X \ ; y \mapsto f(x, y) \in L^1(\mathbb{R}^n)\}.$$

- For $X \subseteq \mathbb{R}^m$ s.a. algebra of "log-s.a. functions"

  $\mathscr{C}(X) := \mathbb{R}$-algebra spanned by $\{g, \log h : \ g, h \in \mathscr{S}(X), \ h > 0\}$

$$\sum_{j \le J} g_j \prod_{k \le K} \log h_{j,k}.$$

Theorem (Lion-Rolin, 1998 + C-Lion-Rolin, 2000).
$f \in \mathscr{S}(X \times \mathbb{R}^n) \Longrightarrow \mathrm{Int}(f)$ is s.a. and $\mathscr{I} \in \mathscr{C}(X)$.

Next question. Stability of the integration process, starting from s.a. functions?

Theorem (Cluckers-D. Miller, 2011).
$f \in \mathscr{C}(X \times \mathbb{R}^n) \Longrightarrow \mathrm{Int}(f) = v^{-1}(0)$ where $v \in \mathscr{C}(X)$ and $\mathscr{I} \in \mathscr{C}(X)$.

# s.a. and oscillatory functions

We now introduce in the game oscillatory finctions (ie $\varphi \neq 0$).

Notation. For $X \subseteq \mathbb{R}^m$ s.a.

- $\mathscr{D}(X) :=$ $\mathbb{C}$-algebra spanned by $\mathscr{C}(X)$ and $\left\{e^{\mathrm{i}\varphi(x)} : \varphi \in \mathscr{S}(X)\right\}$.

Question. $f \in \mathscr{D}(X \times \mathbb{R}^n) \overset{?}{\Rightarrow} \mathscr{I} \in \mathscr{D}(X)$.

**NO**: $\mathscr{D}$ is not stable under integration.

Example. If $\mathscr{D}$ was stable by $\mathscr{I}$ then $e^{-|y|} = \widehat{f}(y) \in \mathscr{D}$, $f(x) = \frac{2}{1+4\pi^2x^2}$. But functions in $\mathscr{D}$ have convergent asymptotic developments of the form (by s.a. preparation)

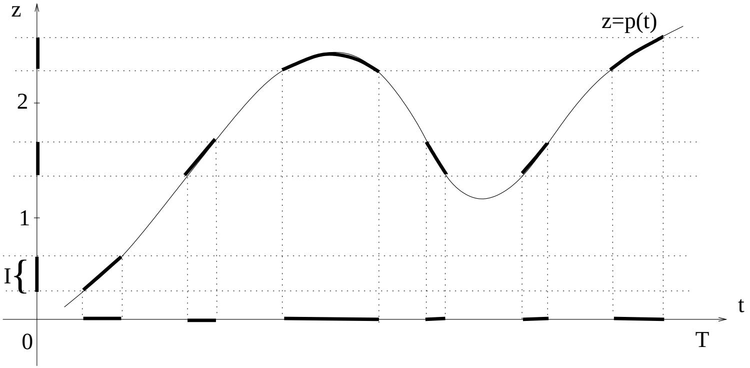
$$e^{-|y|} = \sum_{n \geq m} S_n(y) y^{r_n} \log^{s_n} y, \quad r_n, s_n \in \mathbb{Q}, \quad r_n, s_n \searrow$$

where $S_n$ is a linear combination of $J$ terms $e^{ip_j(y^{1/d})}$, $p_j$ polynomial. This implies $S_m(y) \underset{x \to +\infty}{\longrightarrow} 0$. Contradiction, because:

Fact. $S_m \not\equiv 0$ then $S_m(y)$ does not go to 0 when $y \to +\infty$, but oscillates.

Definition (H. Weyl). A map $(p_j)_{j\leq J} : \mathbb{R} \to \mathbb{R}^J$ is **continuously uniformly distributed mod** 1 (**c.u.d. mod. 1**) if:

$$\forall I \subset [0,1[^J, \quad \lim_{T\to+\infty} \frac{\mathrm{Vol}\{t \in [0,T];\ \{p(t)\} \in I\}}{T} = \mathrm{Vol}(I).$$



## c.u.d. mod 1 maps

Proposition.

- $p_j$ distinct polynomials, $p_j(0) = 0 \Longrightarrow (p_j(y^{1/d}))_{j \leq J}$ is c.u.d. mod. 1,
- $S(y)$ linear comb. of $e^{ip_j(y^{1/d})} \Longrightarrow \exists \varepsilon > 0$ s.t. $\displaystyle\int_{|S| \geq \varepsilon} \frac{\mathrm{d}y}{y} = +\infty$.

$\Longrightarrow$ Fact from the previous slide:

$S_m(y) = \sum_{j \leq J} c_j e^{ip_j(y^{1/d})}$ does not go to 0 when $y \to +\infty$. $\square$

# Uniform c.u.d. mod 1 maps

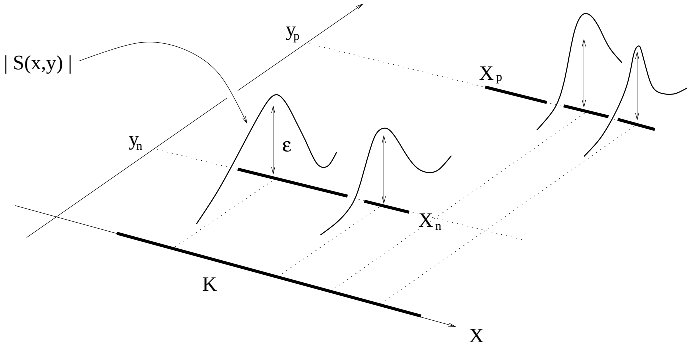
Definition. A family of maps $(p_j(x,y))_{j\leq J} : X \times \mathbb{R} \to \mathbb{R}^J$ is **c.u.d. mod** 1 if:

$$\forall I \subset [0,1[^J, \quad \lim_{T\to+\infty} \sup_{x\in X} \frac{\mathrm{Vol}\{t \in [0,T];\ \{p(x,t)\} \in I\}}{T} = \mathrm{Vol}(I).$$

**Proposition.** Denoting $S(x,y) := \sum_{j \le J} f_j(x) e^{i p_j(x,y)}$

- For $p_j(x,y))_{j \le J}$ c.u.d. mod. 1

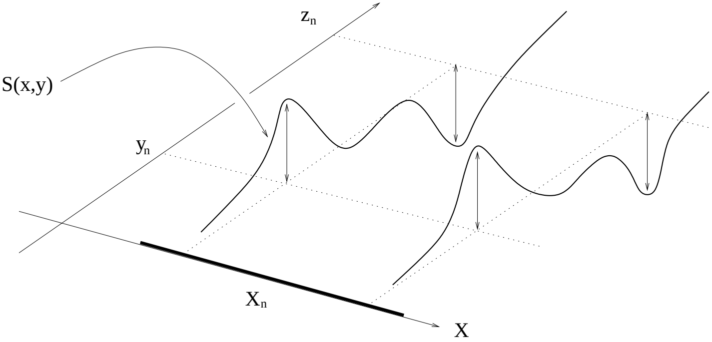
$$\left.\begin{array}{l} \exists \varepsilon > 0, \exists \alpha > 0, \exists K \text{ compact } \subset X, \\ \exists\ X_n \subset K,\ \mathrm{Vol}(X_n) > \alpha, \exists\ y_n \to +\infty, \end{array}\right\} \Longrightarrow \forall x \in X_n,\ |S(x,y_n)| \ge \varepsilon.$$



Proposition. Denoting $S(x,y) := \sum_{j \leq J} f_j(x) e^{ip_j(x,y)}$,

- $p_j(x,y))_{j \leq J}$ c.u.d. mod. 1 $\Longrightarrow$

$$\left.\begin{array}{l} \exists \varepsilon > 0, \exists \alpha > 0, \exists K \text{ compact } \subset X, \\ \exists\ X_n \subset K,\ \mathrm{Vol}(X_n) > \alpha, \\ \exists\ y_n \to +\infty, \exists\ z_n \to +\infty \end{array}\right\} \Longrightarrow \forall x \in X_n,\ |S(x,y_n) - S(x,z_n)| \geq \varepsilon.$$



## Transcendental elements

Find out functions to add to $\mathscr{D}$ to describe the smallest algebra stable by integration and containing $\mathscr{D}$.

$$\boxed{\gamma_{h,\ell}(x) = \int_{\mathbb{R}} h(x,t) \log^{\ell}|t| e^{\mathrm{i}t} dt,}$$

$\ell \in \mathbb{N},\ h \in \mathscr{S}(X \times \mathbb{R}),\ h(x,\cdot) \in L^1(\mathbb{R})$.

Notation. For $X$ s.-a.

- $\mathscr{E}(X) := \mathscr{D}(X)$-*module* spanned by $\{\gamma_{h,\ell}\}_{h,\ell}$.

Stability Theorem. $f \in \mathscr{E}(X \times \mathbb{R}^n) \Longrightarrow$ there exist $h, F \in \mathscr{E}(X)$ s.t.

1. $\mathrm{Int}(f) = h^{-1}(0)$,
2. $\forall x \in \mathrm{Int}(f), \quad F(x) = \int_{\mathbb{R}^n} f(x,y)\,\mathrm{d}y.$

Corollary. *$\mathscr{E}(X)$ is a $\mathbb{C}$-algebra.*

Proof of the Corollary. Fubini:

$\gamma_{h,\ell}(x) \cdot \gamma_{h',\ell'}(x) = \iint_{\mathbb{R}^2} h(x,t) \cdot h'(x,t') \cdot \log^{\ell}|t| \cdot \log^{\ell'}|t'|\, e^{\mathrm{i}(t+t')}\ \mathrm{d}t\mathrm{d}t'$
$\in \mathscr{I}(\mathscr{D}) = \mathscr{E}(X)$ by the Stability Theorem. $\square$

Corollary.

1. $\mathscr{E}$ *is the smallest collection of* $\mathbb{C}$*-algebras containing* $\mathscr{S} \cup \left\{e^{i\varphi} : \varphi \in \mathscr{S}\right\}$ *and stable under integration.* $\square$
2. *In particular* $\mathscr{E}$ *is stable under Fourier transform.* $\square$

# Generators of $\mathscr{E}$

Rem. An element of $\mathscr{E}(X \times \mathbb{R}^n)$ is a finite sum of ***generators*** of type

$$\boxed{T(x,y) = f(x,y) \cdot e^{\mathrm{i}\varphi(x,y)} \cdot \gamma(x,y)} \text{ where}$$

$$f \in \mathscr{C}(X \times \mathbb{R}^n), \ \varphi \in \mathscr{S}(X \times \mathbb{R}^n) \text{ et } \gamma(x,y) = \int_{\mathbb{R}} h(x,y,t) \log^\ell |t| e^{\mathrm{i}t} \, \mathrm{d}t.$$

Def. A generator $T(x,y) \in \mathscr{E}(X \times \mathbb{R}^n)$ is ***super-integrable*** when

$$y \longmapsto |f(x,y)| \int_{\mathbb{R}} \left| h(x,y,t) \log^\ell |t| \right| \, \mathrm{d}t \in L^1(\mathbb{R}^n).$$

Proposition. *$T \in \mathscr{E}(X \times \mathbb{R}^n)$ super-integrable $\Longrightarrow \mathscr{I}_T \in \mathscr{E}(X)$.*

Proof. By Fubini:

$$\int_{\mathbb{R}^n} T(x,y) \, \mathrm{d}y = \iint_{\mathbb{R}^{n+1}} f(x,y) \, h(x,y,t) \log^\ell |t| \, e^{\mathrm{i}(t+\varphi(x,y))} \, \mathrm{d}y\mathrm{d}t.$$

Hence we assume $T = f(x,y) \, e^{\mathrm{i}\varphi(x,y)} \in \mathscr{D}(X \times \mathbb{R}^n)$.

On cells, change of variables : $z = \varphi(x,y)$ and prepare w.r.t. $z$ to obtain a factor $\gamma$. □

# Key step: splitting elements of $\mathscr{E}(X\times\mathbb{R})$

Splitting Lemma. *For $f \in \mathscr{E}(X\times\mathbb{R})$, on cells of $X \times \mathbb{R}$, there exist two finite sets of indices $J^{\mathrm{Super}}, J^{\mathrm{Bad}} \subseteq \mathbb{N}$ and generators $S_j, B_j$ s.t.*

$$f = \sum_{j\in J^{\mathrm{Super}}} S_j + \sum_{j\in J^{\mathrm{Bad}}} B_j,$$

- $S_j$ *super-integrable,*
- $B_j = f_j(x) y^{r_j} \log^{s_j} y \; e^{ip_j(x, y^{1/d})}$, $p_j$ *polynomials,* $\quad r_j \geq -1$

Proof of the splitting Lemma. Expand $h$ as a power series in $y^{1/d}$, then integrate by parts several times to create $S_j$ and $B_j$. $\quad\square$

Proof of the Stability Theorem.

$$\text{Fact}: x \in \mathrm{Int}\left(\sum\nolimits_{j\in J^{\mathrm{Bad}}} B_j, X\right) \implies \forall j \in J^{\mathrm{Bad}}, f_j(x) = 0.$$

$$\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y = \sum\nolimits_{j\in J^{\mathrm{Super}}} \int_{\mathbb{R}} S_j \,\mathrm{d}y \in \mathscr{E}(X).$$

The case $n > 1$ by induction on $n$ and Fubini. $\quad\square$

Fact: $x \in \operatorname{Int}\left(\sum_{j \in J^{\mathrm{Bad}}} B_j, X\right) \Longrightarrow \forall j \in J^{\mathrm{Bad}}, f_j(x) = 0.$

Proof. $B_j(y) = y^{r_j} \log^{s_j} y f_j(x) e^{\mathrm{i} p_j\left(y^{1/d}\right)}$, $p_j$ distinct polynomials.

Let $S(y) = \sum_{j \in J^{\mathrm{Bad}}} f_j(x) e^{\mathrm{i} p_j\left(y^{1/d}\right)}$. Since $y^{r_j} (\log y)^{s_j} \geq y^{-1}$, one has

$$\int_{\mathbb{R}+} \left| \sum_{j \in J^{\mathrm{Bad}}} B_j(y) \right| \mathrm{d}y \geq \int_{\mathbb{R}+} \frac{1}{y} |S(y)| \, \mathrm{d}y \geq \varepsilon \int_{|S| \geq \varepsilon} \frac{\mathrm{d}y}{y} = +\infty. \quad \square$$

# $L^p$-completeness and $L^2$-Fourier transform

"$L^p$-completeness" Theorem. *For* $p \geq 1$, $(f_y(x))_{y\in\mathbb{R}} \in \mathscr{E}(X \times \mathbb{R})$ *Cauchy in* $L^p(X) \Longrightarrow f_y \xrightarrow[y\to+\infty]{\|\ \|_p} f \in \mathscr{E}(X)$.

Corollary. $\mathscr{F} : \mathscr{E} \cap L^2 \xrightarrow{isom} \mathscr{E} \cap L^2$. □

Proof of the $L^p$-completeness Theorem.

$(f_y(x))_{y\in\mathbb{R}}$ Cauchy $\Longrightarrow$ there exists $(y_n)_{n\in\mathbb{N}}$ s.t. $f_{y_n} \xrightarrow[n\to+\infty]{p.p.} f \in L^p(X)$.

We prove that there exists $g \in \mathscr{E}(X)$ s.t. $f_y \xrightarrow[y\to+\infty]{p.p.} g$.

One assume $X$ is a cell and

$$f_y(x) = \sum_{j\leq J} y^{r_j} \log^{s_j} y f_j(x) e^{\mathrm{i}p_j(x,y^{1/d})},$$

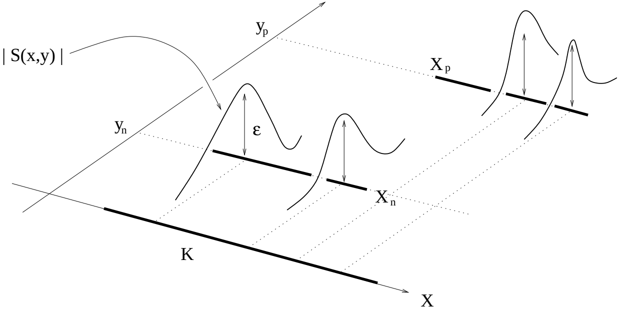
$r_j \geq -1$, $f_j \in \mathscr{E}(X)$, $p_j(x,y)$ polynomials.

Interesting case $\mathbf{r} \geq 0$ or $\mathbf{s} \geq 0$. , where $(\mathbf{r},\mathbf{s}) := \max_{\text{lex}}(r_j, s_j) \geq 0$.

$$f_y(x) = y^{\mathbf{r}} \log^{\mathbf{s}} y \left[ \mathbf{S}(x,y) + \sum_{(r_j,s_j)\neq(\mathbf{r},\mathbf{s})} y^{r_j - \mathbf{r}} \log^{s_j - \mathbf{s}} y \; S_j(x,y) \right],$$

$$\mathbf{S}(x,y) = \sum_{(r_j,s_j)=(\mathbf{r},\mathbf{s})} f_j(x) e^{\mathrm{i}p_j(x,y^{1/d})}.$$

$(y \mapsto p_j(x, y^{1/d}))_{(r_j,s_j)=(\mathbf{r},\mathbf{s})}$ c.u.d. mod. 1 $\implies$

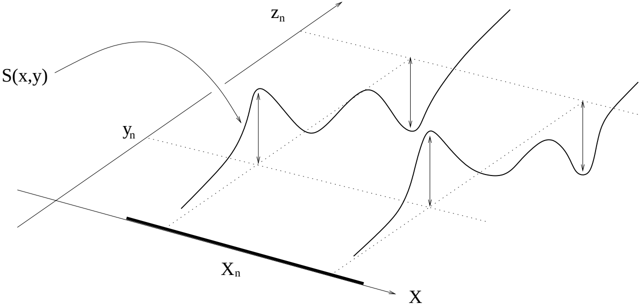



$\implies \mathbf{r} = \mathbf{s} = 0.$

$$f_y(x) = \mathbf{S}(x,y) + \sum_{(r_j,s_j)\neq(\mathbf{r},\mathbf{s})} y^{r_j} \log^{s_j} y \; S_j(x,y),$$

$$\mathbf{S}(x,y) = \sum_{(r_j,s_j)=(\mathbf{r},\mathbf{s})} f_j(x) e^{\mathrm{i}p_j(x,y^{1/d})}.$$

$(y \mapsto p_j(x, y^{1/d}))_{(r_j,s_j)=(\mathbf{r},\mathbf{s})}$ c.u.d. mod. 1 $\implies$




$\implies (f_y)_y$ cannot be Cauchy : contradiction. □