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Multiplicity of complex analytic sets and bilipshitz maps

I want to dedicate this work to the memory of my grandfather, Charles Laure.

Abstract.

We show that the local multiplicity of complex analytic sets is preserved by bilipschitz maps with lipschitz constants satisfying a certain inequality in which only the multiplicities in question and the local dimension of the analytic sets play a part. We also show the continuity of the density function along strata of a lipschitz trivial stratification of a subanalytic set and thus the constancy of the multiplicity along strata of a lipschitz trivial stratification of a complex analytic set.

1. INTRODUCTION

Let X be a complex analytic closed set of dimension d of $\mathbb{C}^n$, and $x \in X$.

The multiplicity of X at x is defined to be the multiplicity of the local ring $\mathcal{O}_{x,X}$; it measures the singularity of X in x and equals 1 if and only if x is a regular point of X.

Geometrically, $\mathfrak{M}_x(X)$ is the number of points of the general fiber of a generic projection onto $\mathbb{C}^d$, defined in a neighbourhood of x (see [Dra] for the equivalence between the two definitions).

If X is a reduced hypersurface $h^{-1}(0)$, then $h(y+x) = \sum_{j=\mathfrak{M}_x(X)}^{\infty} h_j(y)$ where $h_j(y)$ is a homogeneous polynomial in y, $h_{\mathfrak{M}_x(X)}$ is of degree $\mathfrak{M}_x(X)$ and for $j > \mathfrak{M}_x(X)$, h_j is null or of degree j.

In 1971 Zariski asked the following question which is still open: is the multiplicity of hypersurfaces a topological invariant ([Zar 1]) ? That is to say, if (X_1, x_1) et (X_2, x_2) are two germs of complex hypersurfaces represented by X_1 and X_2 defined in the open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathbb{C}^n$ and if there exists $f : (\mathcal{U}_1, X_1, x_1) \longrightarrow (\mathcal{U}_2, X_2, x_2)$ a homeomorphism which sends $\mathcal{U}_1$ onto $\mathcal{U}_2$, X_1 onto X_2 and x_1 to x_2, is $\mathfrak{M}_{x_1}(X_1)$ equal to $\mathfrak{M}_{x_2}(X_2)$?

It is known that if $n = 2$ ([Zar 2]), or $\mathfrak{M}_{x_1}(X_1) = 1$ ([ACa] and [Lê]), or $\mathfrak{M}_{x_1}(X_1) = 2$ and $n = 3$ ([Na]), or f is a $\mathcal{C}^1$ diffeomorphism (cf [Eph] or [Tr]) the answer is positive .

A recent result of Risler and Trotman ([Ris-Tr]) says that if (X_1, x_1) and (X_2, x_2) are two complex hypersurface germs right-left-equivalent by bilipschitz homeomorphisms, then $\mathfrak{M}_{x_1}(X_1) = \mathfrak{M}_{x_2}(X_2)$.

More precisely, if a representative of (X_1, x_1) is $X_1 = f_1^{-1}(0)$ and a representative of (X_2, x_2) is $X_2 = f_2^{-1}(0)$, and if there are positive constants A, B, C and C' such that:

$$C'|x-y| \leq |f(x) - f(y)| \leq C|x-y| \;\; \forall x \in \mathcal{U}_1 \, , \, \forall y \in X_1$$

$$\textit{and} \;\; 0 < A \leq \frac{|f_2 \circ f|}{|f_1|} \leq B < \infty \;\; \text{on } \mathcal{U}_1 \setminus X_1$$

then the multiplicity of X_1 at x_1 equals the multiplicity of X_2 at x_2.

The Zariski problem has a negative answer for analytic sets of codimension greater than 1; there are curves in $\mathbb{C}^3$ which have the same topological type but distinct multiplicities (cf. [Gau-Lip] for example and note that these curves have not the same local bilipschitz type).

However Gau and Lipman, in generalizing Ephraim's result, showed that if (X_1, x_1) and (X_2, x_2) are two germs of analytic sets of dimension $d < n$ in $\mathbb{C}^n$ with the same differential type (f and f^{-1} are differentiable homeomorphisms in x_1 and x_2), then $\mathfrak{M}_{x_1}(X_1) = \mathfrak{M}_{x_2}(X_2)$.

2. RESULTS

We give here a non-differential type result about complex analytic sets of all codimensions: for a bilipschitz homeomorphism between two complex analytic set germs, we state a sufficient and explicit condition for equimultiplicity.

Remark 1. — We do not suppose that f is defined on a neighbourhood of x_1 in the ambient space $\mathbb{C}^n$ as do Gau and Lipman ([Gau-Lip]), but only on a neighbourhood of x_1 in X_1.

Theorem 1. — *Given (X_1, x_1) and (X_2, x_2) two complex analytic germs of $\mathbb{C}^n$ of dimension $d \leq n$, $M = max(\mathfrak{M}_{x_1}(X_1), \mathfrak{M}_{x_2}(X_2))$ and $f : (X_1, x_1) \longrightarrow (X_2, x_2)$ a bilipschitz homeomorphism such that:*

$$\frac{1}{C'}|x-y| \leq |f(x) - f(y)| \leq C|x-y| \;\; \forall x, y \;\; \textit{near } x_1 \;\; \textit{in } X_1$$

$$\textit{and} \;\; (1 \leq) C'C \leq (1 + \frac{1}{M})^{\frac{1}{2d}}$$

then: $\mathfrak{M}_{x_1}(X_1) = \mathfrak{M}_{x_2}(X_2)$.

Now we call a map $F : (X, x) \times [0,1] \longrightarrow \mathbb{R}^n$ a bilipschitz deformation of the germ (X, x) if for all $t \in [0,1]$, $(F_t(X), F_t(x))$ is a germ, F_0 is the identity of (X, x) and $F_t : (X, x) \longrightarrow (F_t(X), F_t(x))$ (resp. $F_t^{-1} : (F_t(X), F_t(x)) \longrightarrow (X, x)$) is a lipschitz homeomorphism with lipschitz constant $C_t = Lip(F_t) = sup\{\frac{|F_t(z) - F_t(y)|}{|z-y|}; \; z \neq y \in X\}$ (resp. $C'_t = Lip(F_t^{-1}) = sup\{\frac{|F_t^{-1}(z) - F_t^{-1}(y)|}{|z-y|}; \; z \neq y \in F_t(X)\}$) tending to

1 when t tends to 0 (F_t means the restriction of F to $(X,x) \times \{t\}$). One has the following:

Theorem 2. — *Small enough bilipschitz deformations of complex analytic sets are equimultiple: there exists $0 < t_0 < 1$ such that the family of germs $(F_t(X), (F_t(x))_{(0 \leq t < t_0)}$ is equimultiple.*

Remark 2. — A well known result (proved by Hironaka in [Hi], or by Teissier in [Tei] and by Henry and Merle in [Hen-Me]) states the equimultiplicity of a reduced complex analytic set along a stratum of any one of its (b)-regular stratifications. Since a lipschitz stratification(†) of a subanalytic set is obviously (w)-regular, it also satisfies the Kuo ratio test, and thus by [Kuo] is (b)-regular. So one can deduce from Hironaka's theorem the constancy of the multiplicity of a complex analytic set along a stratum of a lipschitz stratification. Theorem 2 gives a direct simple proof of this constancy: along a stratum S of a lipschitz stratification of X any germ (X,x) is a bilipschitz deformation of any other germ (X,y), with y close enough to x; by theorem 2 $(\mathfrak{M}_x(X))_{(x \in S)}$ is locally constant, and thus constant.

Remark 3. — If X is a germ of a reduced complex analytic set and $Y \subset X$ a smooth complex analytic variety such that X is lipschitz equisaturated along Y(‡), it is known that X is equimultiple along Y. By Theorem 2 we find this result again: the equisaturation of X along Y implies the lipschitz triviality of X along Y ([Pha-Tei], theorem 4 or [Pha]), obtained by integration of a lipschitz vector field.

Remark 4. — It seems that there is no known example of a topologically trivial complex analytic stratification of a complex hypersurface which does not admit a lipschitz deformation along its strata. For the non hypersurface case, the referee suggested this example of a Whitney-regular stratification, which is not bilipschitz trivial: $X_t = \{(x,y,z) \in \mathbb{C}^3 , z = x^2, y = 0\} \cup \{(x,y,z) \in \mathbb{C}^3 , y = tx, z = 0\}$.

Remark 5. — The proof of theorem 2 actually shows that the density function (defined below) is continuous along a lipschitz deformation of a subanalytic set.

3. PROOF OF THE THEOREMS

The proofs of the theorems use a result of Draper ([Dra]): the multiplicity $\mathfrak{M}_x(X)$ is the density of X at x.

Let $B_{(x,r)}$ be the closed ball of $\mathbb{R}^{m=2n} = \mathbb{C}^n$ centred in x with radius $r > 0$, let $s \leq m$, be a positive integer, $\mathcal{H}^s$ the s-dimensional Hausdorff measure of $\mathbb{R}^m$ (cf [Fe]). We consider the ratio (classical in geometric measure theory and first studied by P. Lelong in [Lel]):

$$\rho_{(x,r,s)} = \frac{\mathcal{H}^s(B_{(x,r)} \cap X)}{\mu_s r^s}$$

(where μ_s is the s-dimensional volume of the unit ball of $\mathbb{R}^s$). If $\rho_{(x,r,s)}$ tends to a limit when r tends to 0, this limit is called the density of X at x and is denoted $\Theta_s(X,x)$.

(†)The lipschitz stratifications have been defined by T. Mostowski in [Most] and developed by A. Parusinski in [Pa1], [Pa2] and [Pa3].

(‡)this notion of equisingularity has been introduced by B. Teissier and F. Pham in [Pha-Tei] and [Pha].

Theorem 3 [Dra]. — *Let X be a complex analytic set of complex dimension d and x a point of X, then $\Theta_{2d}(X,x) = \mathfrak{M}_x(X)$.*

The proofs of theorems 1 and 2 use a geometric measure lemma, which seems not to have been stated explicitly before.

Lemma. — *Let (X_1,x_1) and (X_2,x_2) be two germs of subanalytic sets of $\mathbb{R}^m$ of dimension δ, (so $\Theta_\delta(X_1,x_1)$ and $\Theta_\delta(X_2,x_2)$ exist by [Kur-Ra]), and let $f : (X_1,x_1) \to (X_2,x_2)$ be a germ of a bilipschitz homeomorphism, C be the Lipschitz constant of f, C' the Lipschitz constant of f^{-1}. Then:*

$$\frac{1}{(CC')^\delta}\Theta_\delta(X_1,x_1) \leq \Theta_\delta(X_2,x_2) \leq (CC')^\delta \Theta_\delta(X_1,x_1).$$

Proof. For all (x,y) in $X \times X$:

$$\frac{1}{C'}|x-y| \leq |f(x)-f(y)| \leq C|x-y|. \qquad (i)$$

And thus we have: $B_{(x_2,\frac{r}{C'})} \cap X_2 \subseteq f(B_{(x_1,r)} \cap X_1) \subseteq B_{(x_2,Cr)} \cap X_2$, and the following inequalities:

$$\mathcal{H}^\delta(B_{(x_2,\frac{r}{C'})} \cap X_2) \leq \mathcal{H}^\delta(f(B_{(x_1,r)} \cap X_1)) \leq \mathcal{H}^\delta(B_{(x_2,Cr)} \cap X_2)$$

$$\frac{1}{C'^\delta}\frac{\mathcal{H}^\delta(B_{(x_2,\frac{r}{C'})} \cap X_2))}{\mu_\delta(\frac{r}{C'})^\delta} \leq \frac{\mathcal{H}^\delta(f(B_{(x_1,r)} \cap X_1))}{\mu_\delta r^\delta} \leq C^\delta\frac{\mathcal{H}^\delta(B_{(x_2,Cr)} \cap X_2))}{\mu_\delta(Cr)^\delta}.$$

Hence $$\frac{1}{C'^\delta}\Theta_\delta(X_2,x_2) \leq \liminf_{r\to 0}\frac{\mathcal{H}^\delta(f(B_{(x_1,r)} \cap X_1))}{\mu_\delta r^\delta} \qquad (ii)$$

and $$\limsup_{r\to 0}\frac{\mathcal{H}^\delta(f(B_{(x_1,r)} \cap X_1))}{\mu_\delta r^\delta} \leq C^\delta\Theta_\delta(X_2,x_2) \qquad (ii\,bis).$$

We now compare $\liminf_{r\to 0}\frac{\mathcal{H}^\delta(f(B_{(x_1,r)} \cap X_1))}{\mu_\delta r^\delta}$, $\limsup_{r\to 0}\frac{\mathcal{H}^\delta(f(B_{(x,r)} \cap X_1))}{\mu_\delta r^\delta}$ and $\Theta_\delta(X_1,x_1)$. The inequalities *(i)* give us:

$$\frac{1}{C'^{\delta}}\mathcal{H}^{\delta}(B_{(x_1,r)}\cap X_1) \leq \mathcal{H}^{\delta}(f(B_{(x_1,r)}\cap X_1)) \leq C^{\delta}\mathcal{H}^{\delta}(B_{(x_1,r)}\cap X_1).$$

So we have

$$\frac{1}{C'^{\delta}}\Theta_{\delta}(X_1,x_1) \leq \liminf_{r\to 0}\frac{\mathcal{H}^{\delta}(f(B_{(x_1,r)}\cap X_1))}{\mu_{\delta}r^{\delta}} \qquad (iii)$$

and $$\limsup_{r\to 0}\frac{\mathcal{H}^{\delta}(f(B_{(x_1,r)}\cap X_1))}{\mu_{\delta}r^{\delta}} \leq C^{\delta}\Theta_{\delta}(X_1,x_1) \qquad (iii\,bis).$$

Now (1) follows easily from $(ii), (ii\,bis), (iii)$, and $(iii\,bis)$. □

The lemma and theorem 3 make the proofs of theorem 1 and 2 a simple computation.

Proof of theorem 1. Suppose $\Theta_{2d}(X_1,x_1) < \Theta_{2d}(X_2,x_2)$. By the lemma we have:

$$0 < \Theta_{2d}(X_2,x_2)-\Theta_{2d}(X_1,x_1) \leq ((CC')^{2d}-1)\Theta_{2d}(X_1,x_1) < ((CC')^{2d}-1)M.$$

But the hypothesis of theorem 1 gives :

$$0 < \Theta_{2d}(X_2,x_2) - \Theta_{2d}(X_1,x_1) < 1,$$

By theorem 3 the last inequality is absurd, and finally $\Theta_{2d}(X_1,x_1) = \Theta_{2d}(X_2,x_2)$; completing the proof, again by theorem 3. □

Proof of theorem 2. The lemma gives us:

$$(\frac{1}{(C_tC_t')^{2d}}-1)\Theta_{2d}(X,x) \leq \Theta_{2d}(F_t(X),F_t(x))-\Theta_{2d}(X,x) \leq ((C_tC_t')^{2d}-1)\Theta_{2d}(X,x).$$

Since $C_tC_t' \to 1$ when $t \to 0$, for small enough $t > 0$: $|\Theta_{2d}(F_t(X),F_t(x)) - \Theta_{2d}(X,x)| < 1$. But by theorem 3, $\Theta_{2d}(X,x) = \mathfrak{M}_x(X)$ and $\Theta_{2d}(F_t(X),F_t(x)) = \mathfrak{M}_{F_t(x)}(F_t(X))$ are integers; for small enough $t > 0$ one thus has: $\mathfrak{M}_x(X) = \mathfrak{M}_{F_t(x)}(F_t(X))$. □

Acknowledgement. — *I wish to thank D. Trotman and B. Teissier for their suggestions.*

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