

ROTATION OF TRAJECTORIES OF LIPSCHITZ VECTOR FIELDS

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Abstract

We prove that the rotation in time T of a trajectory of a K -Lipschitz vector field in \mathbb{R}^n around a given point (stationary or non-stationary) is bounded by $A + BKT$ with A, B absolute constants. In particular, trajectories of a Lipschitz vector field in finite time cannot have an infinite rotation around a given point (while trajectories of a C^∞ vector field may have an infinite rotation around a straight line in finite time). The bound above extends to the mutual rotation of two trajectories (for the time intervals T and T' , respectively) of a K -Lipschitz vector field in \mathbb{R}^3 : this rotation is bounded from above by the quantity $CK \min(T, T') + DK^2TT'$.

1. Introduction

In this paper we investigate the tameness of a geometric behavior of trajectories of vector fields: the rotation of such a trajectory around a point or the mutual rotation of two trajectories.

Of course, a lot of results have been published on the geometry of solutions of differential equations and trajectories of vector fields, and we simply cannot give an extensive review or even a bibliography on the subject. Nevertheless, we have at our disposal only few global theories and general results concerning the tameness of trajectories. In this very short introduction we just want to focus on two of them.

In Gabriellov-Hovanskii's theory of Pfaffian sets, a Pfaffian function f on an open set $\mathcal{U} \subset \mathbb{R}^n$ is defined as a function that can be written in the following way: $f(x) = P(x, c_1(x), \dots, c_m(x))$, where P is a polynomial and the c_i 's are analytic solutions of the polynomial triangular differential system:

$$(*) \quad Dc_i(x) = \sum_{j=1}^m P_{i,j}(x, c_1(x), \dots, c_i(x)) dx_j, \quad i \in \{1, \dots, m\}.$$

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Then, if we consider the Pfaffian structure, that is to say the smallest structure containing the semi-Pfaffian sets, it has been proved in [Wi], using a Bezout type theorem of Hovanskii ([Ho 1], [Ho 2]), that this structure is o-minimal (see [Co], [Dr], [Dr-Mi], [Sh]): for short, the number of connected components of sets in such a structure is finite, and consequently, no Pfaffian curve (nor set) may infinitely oscillate or spiral around a point and two such curves have bounded mutual rotation.

This theory presents a large class of tame objects coming from differential equations, but when we deal with vector fields, the only way to a priori be sure that the trajectories belong to this category is to assume that the trajectory is a Pfaffian function itself, satisfying equation (*). We deduce from this assumption that the field depends only on one variable. A too restrictive hypothesis that obviously not allows a wild behavior for trajectories.

On the other hand, starting with a given vector field and studying the local geometry of its trajectories in a neighborhood of one of its singularities, we know that this geometry is tame, following [Ku-Mo] and [Ku-Mo-Pa], provided that the field is an analytic gradient vector field: the rotation of a trajectory of an analytic gradient vector field around one of its singularities is finite. As a consequence, the limit of the secant lines to the trajectory exists (Thom's Gradient Conjecture). Let us notice that we do not know whether a trajectory of an analytic gradient vector field lies in some o-minimal structure, although we know that trajectories of the gradient of a function definable in a given o-minimal structure have finite length (see [Ku]). On the local behavior of trajectories we would like to refer to the number of deep results produced by the *Spanish-Dijon School*, and specially the most recent ones: [Ca-Mo-Ro], [Ca-Mo-Sa 1], [Ca-Mo-Sa 2], [Ca-Mo-Sa 2], [Bl-Mo-Ro].

The aim of this paper is to give general results about rotation of trajectories of a vector field, with no restrictive assumptions on the nature of the field, besides the Lipschitz property which is a minimal hypothesis for the existence of trajectories.

Of course, in this direction, we cannot hope to treat infinite time phenomena, as it is done in [Ku-Mo-Pa] and the Pfaffian theory because it is well-known that complete trajectories of polynomial vector fields in the plane may spiral around a singularity of the field (see, for instance, Remark of Section 3.1). We thus obtain our bounds for trajectories defined on a finite time interval. For other bounds obtained in the same spirit, the reader may report to [Gr-Yo], [Ho-Ya], [No-Ya 1], [No-Ya 2], [Ya]. Of course, we can then let the time go to infinity in our bounds in order to get asymptotic bounds. In this direction, one direct consequence of the existence of our bounds is an explicit upper

bound for the so-called asymptotic Hopf invariant (Theorem 3.12) defined by Arnold in [Ar] in order to compute the minimum energy of fields obtained from a given divergence-free field under the action of volume-preserving diffeomorphisms. Another immediate consequence given in this paper is a logarithmic bound for the asymptotic local rotation at singular points of analytic vector fields (Theorem 3.13).

The paper is organized as follows.

In Section 2, we introduce and compare some notions of absolute and topological rotation for trajectories around an affine subspace in \mathbb{R}^n or for two trajectories around each others in \mathbb{R}^3 .

In Section 3, we first notice that the rotation of any trajectory of a Lipschitz vector field around its stationary points is bounded in terms of the Lipschitz constant and the time elapsed only (Proposition 3.1). The same is true for the rotation of any trajectory around a linear invariant subspace of the field (Proposition 3.2). Moreover, while the rotation velocity of a trajectory around a *non-stationary point* of the field may tend to infinity, we prove our first main result: the “total” rotation around such a point is still bounded in terms of the Lipschitz constant and the time interval (Theorem 3.8). Our second main result is a consequent of the first one: we give a uniform bound for the mutual rotation of any two trajectories of a given Lipschitz vector field in terms of the time interval and the Lipschitz constant (Theorem 3.9). In contrast, we provide an easy example showing that the rotation of a trajectory of C^∞ vector field around a non-invariant subspace may be infinite in finite time (Example 3.3). The end of the paper is devoted to the two direct applications mentioned above (Theorems 3.12 and 3.13).

We would like to thank the referee for finding a gap in our initial arguments and pointing out a possible way to close it.

2. Definition of signed and absolute rotation

2.1. Rotation around an affine subspace. First of all, we define an absolute rotation of the curve γ in \mathbb{R}^n around the origin $0 \in \mathbb{R}^n$. Assuming that γ does not pass through the origin, we can define the spherical image of γ as the curve σ in the unit sphere S^{n-1} in the following way:

$$\sigma(t) = \frac{\gamma(t)}{\|\gamma(t)\|}.$$

Definition. *The absolute rotation $R_{\text{abs}}(\gamma, 0)$ of the curve γ around the origin is the length of σ , the spherical image of γ in the unit sphere S^{n-1} .*

We have the following lemma (easy to check) which relates the rotation to the spherical part of the velocity of the curve:

Lemma 2.1. Let $\gamma'(t) = \frac{d\gamma(t)}{dt}$ be the velocity vector of $\gamma : I \rightarrow \mathbb{R}^n$, and let $\gamma'_r(t)$ and $\gamma'_s(t)$ be the radial and the spherical components of this velocity vector. Then the velocity of the spherical blowing-up σ of γ is

$$\sigma'(t) = \frac{\gamma'_s(t)}{\|\gamma(t)\|}.$$

As a consequence, the absolute rotation $R_{\text{abs}}(\gamma, 0)$ is given by the integral

$$R_{\text{abs}}(\gamma, 0) = \int_{t \in I} \frac{\|\gamma'_s(t)\|}{\|\gamma(t)\|} dt.$$

Remark. The absolute rotation $R_{\text{abs}}(\gamma, 0)$ is invariant with respect to the monotone reparametrizations of the curve γ . In the same way we want our results to be about the geometry of the curve and not depending on one of its parametrizations. This is why in what follows we implicitly consider the geometric trajectory $\gamma(I)$ of the injective curve $\gamma : I \rightarrow \gamma(I)$ as the class of $\gamma : I \rightarrow \mathbb{R}^n$ modulo its injective reparametrizations.

For plane curves γ in \mathbb{R}^2 , we can define their “signed rotation” $R(\gamma, 0)$ around the origin, which is, of course, the usual rotation index. Indeed, in this case the unit sphere is a circle. Assuming that the orientation of this circle (and of the curve γ) has been chosen, we can define the signed rotation essentially by the same expression as above:

Definition. The *signed rotation* $R(\gamma, 0)$ of γ around the origin is defined by

$$R(\gamma, 0) = \frac{1}{2\pi} \int_{\gamma} \pm \frac{\|\gamma'_s(t)\|}{\|\gamma(t)\|} dt.$$

Here the sign under the integral is chosen according to the direction of the tangent to the unit circle vector $\gamma'_s(t)$.

Remark. Notice that the normalization by 2π , the unit circle length, which appears here to make the linking number defined below, is an integer. In codimension greater or equal to three, we do not normalize the length of the spherical curves.

An absolute rotation of the curve γ in \mathbb{R}^n around a linear k -dimensional subspace $\mathcal{L} \subset \mathbb{R}^n$ is defined as follows: let \mathcal{L}^\perp denote the orthogonal subspace to \mathcal{L} . Let $\tilde{\gamma}$ be the projection of γ on \mathcal{L}^\perp . Assuming that γ does not touch \mathcal{L} , we get $\tilde{\gamma}$ not passing through the origin in \mathcal{L}^\perp .

Definition. The *absolute rotation* $R_{\text{abs}}(\gamma, \mathcal{L})$ is defined as the absolute rotation of the curve $\tilde{\gamma}$ in \mathcal{L}^\perp around the origin. In the case of a linear subspace \mathcal{L} of codimension 2 in \mathbb{R}^n , a signed rotation $R(\gamma, \mathcal{L})$ of γ around \mathcal{L} is defined as the signed rotation of the curve $\tilde{\gamma}$ in the plane \mathcal{L}^\perp around the origin.

Of course, the absolute rotation always bounds from above the absolute value of the signed one.

For a closed curve γ and for a subspace \mathcal{L} of codimension 2, the signed rotation $R(\gamma, \mathcal{L})$ is an integer, and it is a topological invariant. For γ non-closed, this signed rotation $R(\gamma, \mathcal{L})$ may take non-integer values. However, it is still an invariant of deformations of γ preserving the end points and not touching \mathcal{L} .

2.2. Rotation of two curves in \mathbb{R}^3 . It is well known that the linking number of two closed curves in \mathbb{R}^3 can be defined via an integral expression, the so-called Gauss integral (see, for example, [Ar-Kh], [Du-Fo-No]). This gives us a natural way to define also an absolute and a signed rotation of two curves (closed or non-closed) one around the other.

For non-closed curves, the rotation defined in this (or any other) way cannot be metrically or topologically invariant. But on the other hand, the Gauss integral representation provides a powerful analytic tool for its investigation. In the next subsection, we remind the construction of the Gauss integral and its main properties. Our presentation follows very closely the one given in [Du-Fo-No].

2.2.1. The Linking Coefficient. Consider a pair of smooth, closed, regular directed curves in \mathbb{R}^3 , which do not intersect. We may assume them to be parametrized in the following way: $\gamma_i : I_i \rightarrow \mathbb{R}^3$, $i = 1, 2$, with I_1, I_2 two compact intervals. We denote our geometric curves by γ_1 and γ_2 (instead of $\gamma_1(I_1), \gamma_2(I_2)$).

Definition. The *linking coefficient of the two curves* γ_1, γ_2 is defined in terms of the ‘‘Gauss integral’’ by

$$\{\gamma_1, \gamma_2\} = \frac{1}{4\pi} \int_{t_1 \in I_1} \int_{t_2 \in I_2} \frac{\langle \gamma_1'(t_1) \wedge \gamma_2'(t_2), \gamma_{12}(t_1, t_2) \rangle}{\|\gamma_{12}(t_1, t_2)\|^3} dt_1 dt_2,$$

where $\gamma_{12}(t_1, t_2) = \gamma_2(t_2) - \gamma_1(t_1)$.

Remark. In this definition, the normalization by 4π has to be seen as the normalization by the volume of the unit sphere of \mathbb{R}^3 (see the Remark that follows the proof of Theorem 2.2).

Let us stress that this definition immediately shows that $\{\gamma_1, \gamma_2\}$ does not depend on the parametrization of the curves nor on their rigid transformations. Intuitively speaking the linking coefficient gives the algebraic (i.e., signed) number of loops of one contour around the other. This interpretation is justified by the following result:

Theorem 2.2. *Let γ_1 and γ_2 be two closed curves in \mathbb{R}^3 and assume that $I_1 = [0, 2\pi]$.*

- (i) *The linking coefficient $\{\gamma_1, \gamma_2\}$ is an integer, and is unchanged by deformations of γ_1 and γ_2 , involving no intersection of one curve with the other.*
- (ii) *Let $F : D^2 \rightarrow \mathbb{R}^3$ be a map of the disc D^2 which agrees with $\gamma_1 : t \mapsto \gamma_1(t)$, $0 \leq t \leq 2\pi$ on the boundary $\partial D^2 \simeq S^1 \simeq \frac{[0, 2\pi]}{0 \sim 2\pi}$, and is transversal to the curve $\gamma_2 \subset \mathbb{R}^3$. Then the “topological linking number,” which is the intersection index $F(D^2) \cdot \gamma_2$ (i.e., the number of the intersection points of $F(D^2)$ and γ_2 , counted with the signs reflecting the orientation), is equal to the linking coefficient $\{\gamma_1, \gamma_2\}$.*

Proof. The closed curves $\gamma_i(t)$, $i = 1, 2$, give rise to a 2-dimensional, closed, oriented parametric surface $\gamma_1 \times \gamma_2$ in \mathbb{R}^6 :

$$\gamma_1 \times \gamma_2 : (t_1, t_2) \mapsto (\gamma_1(t_1), \gamma_2(t_2)).$$

Since the curves are non-intersecting, the map $\varphi : \gamma_1 \times \gamma_2 \rightarrow S^2$ given by

$$\varphi(t_1, t_2) = \frac{\gamma_1(t_1) - \gamma_2(t_2)}{\|\gamma_1(t_1) - \gamma_2(t_2)\|}$$

is well defined. An easy geometric consideration shows that the integrand in the Gauss integral is just the Jacobian of the map φ . Therefore, the Gauss integral above is equal to the degree of the map φ . Hence the linking coefficient is, indeed, an integer. Under deformations of the curves $\{\gamma_1, \gamma_2\}$ involving no intersection one with the other, the map φ undergoes a homotopy, so that its degree, and therefore also the linking coefficient, are preserved. Let us stress that in the process of these deformations each of the curves γ_1, γ_2 separately may cross itself in an arbitrary way. Of course, the topological linking number is also preserved by such deformations.

We now prove (ii). If the curves are not linked (i.e., if by means of a homotopy respecting non-intersection, they can be brought to opposite sides of a 2-dimensional plane in \mathbb{R}^3), then it can be verified directly that $\{\gamma_1, \gamma_2\} = \deg \varphi = 0$. In a general case, we can “push” γ_1 along γ_2 in such a way that after this deformation, it comes close to γ_2 only in a neighborhood of exactly one point. Then by applying another deformation (remember that self-intersections of the curves are allowed), we reduce the general case of the problem of calculating the linking coefficient essentially to the following simple situation: the curve γ_2 is a straight line, while γ_1 is a circle, orthogonal to γ_2 and passed several times in the positive or negative direction. Thus, we suppose γ_1 and γ_2 to be given respectively by $\gamma_1(t_1) = (\cos t_1, \sin t_1, 0)$, $0 \leq t_1 \leq 2\pi$ and $\gamma_2(t_2) = (0, 0, t_2)$, $-\infty < t_2 < \infty$. The linking coefficient for these two

curves is

$$\begin{aligned} \{\gamma_1, \gamma_2\} &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{dt_1 dt_2}{(1+t_2^2)^{3/2}} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dt_2}{(1+t_2^2)^{3/2}} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{ch^2(u)} = \frac{1}{2} [th(u)]_{-\infty}^{+\infty} = 1. \end{aligned}$$

Hence, for these two directed curves, the statement (ii) of the Theorem holds. The general result now follows via the deformation described above. q.e.d.

Remark. One can give another proof of Theorem 2.2. As above, we notice that the Gauss integral is equal to the degree of the mapping $\varphi : \gamma_1 \times \gamma_2 \rightarrow S^2$. Now fix a point p in S^2 which is a regular value of φ and consider the projection π along the corresponding line ℓ_p onto the orthogonal plane P_p . The preimages $\varphi^{-1}(p)$ correspond exactly to the crossing points of the plane curves $\pi(\gamma_1)$ and $\pi(\gamma_2)$ in P_p . The sign of the Jacobian of φ at each of the preimages $\varphi^{-1}(p)$ can be computed via the directions of $\pi(\gamma_1)$ and $\pi(\gamma_2)$ at their corresponding crossing point taking into account, which curve is “above” and which is “below”.

Now the degree of the mapping φ is the sum of the signs of the Jacobian of φ over all the preimages $\varphi^{-1}(p)$. On the other hand, the corresponding sum over all the crossing points of $\pi(\gamma_1)$ and $\pi(\gamma_2)$ can be easily interpreted as the topological linking number of γ_1 and γ_2 .

2.2.2. Signed and absolute rotation. As the curves γ_1, γ_2 are not necessarily closed, the Gauss integral can be still computed.

Definition. For two curves γ_1, γ_2 in \mathbb{R}^3 , closed or non-closed, we call the Gauss integral along these curves the *signed rotation of the curves* γ_1 and γ_2 and denote it by $R(\gamma_1, \gamma_2)$. We have

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{t_1 \in I_1} \int_{t_2 \in I_2} \frac{\langle \gamma'_1 \wedge \gamma'_2, \gamma_{12} \rangle}{\|\gamma_{12}\|^3} dt_1 dt_2,$$

where $\gamma_{12} = \gamma_2 - \gamma_1$.

For two curves γ_1, γ_2 , not necessarily closed, the *absolute rotation* of these curves is defined as

$$R_{\text{abs}}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{t_1 \in I_1} \int_{t_2 \in I_2} \frac{|\langle \gamma'_1 \wedge \gamma'_2, \gamma_{12} \rangle|}{\|\gamma_{12}\|^3} dt_1 dt_2.$$

Remarks. The absolute rotation of two curves bounds the absolute value of their signed rotation:

$$|R(\gamma_1, \gamma_2)| \leq R_{\text{abs}}(\gamma_1, \gamma_2).$$

In particular, for γ_1, γ_2 closed, the absolute rotation bounds the absolute value of the linking number $\{\gamma_1, \gamma_2\}$.

For γ_1, γ_2 not closed, $R(\gamma_1, \gamma_2)$ does not need to be an integer anymore. This rotation number also is not invariant under the deformations

of the curves γ_1, γ_2 without crossing one another, even if we assume that the end-points of γ_1 and γ_2 are fixed. Indeed, if we take the curve γ_1 to be a “long” segment of the straight line, and the curve γ_2 to be the unit circle around γ_1 , the computation at the end of the proof of Theorem 2.2 above shows that the signed rotation $R(\gamma_1, \gamma_2)$ is approximately one. On the other hand, we can deform the circle γ_2 with one of its points fixed as follows: we pull it out from the segment γ_1 , and then contract it to the point. The rotation of the deformed curves is zero, so it was not preserved in the process of the deformation.

However, for one of the curves, say γ_1 , closed, we have the following result:

Proposition 2.3. *Let the curve γ_1 be closed. Then the signed rotation $R(\gamma_1, \gamma_2)$ is invariant under the deformations of the curve γ_2 (without crossing γ_1) if the end-points of γ_2 remain fixed.*

Proof. Consider a closed curve $\tilde{\gamma}_2$ obtained from γ_2 by passing it twice in the opposite directions. We have $R(\gamma_1, \tilde{\gamma}_2) = 0$, since the rotation of these two closed curves is invariant under deformation, while $\tilde{\gamma}_2$ can be deformed into the point without crossing γ_1 . Now consider another deformation of $\tilde{\gamma}_2$, where one copy of γ_2 remains fixed, while another copy, $\hat{\gamma}_2$, undergoes a deformation without crossing γ_1 and with the end points fixed. The signed rotation remains zero in this deformation, so $R(\gamma_1, \tilde{\gamma}_2) = R(\gamma_1, \gamma_2) - R(\gamma_1, \hat{\gamma}_2) = 0$. Hence $R(\gamma_1, \hat{\gamma}_2)$ remains the same in the deformation. q.e.d.

Another property which can be obtained by a rather straightforward computation is the following:

Proposition 2.4. *For $\gamma_1 = \mathcal{L}$ a straight line, the signed rotation $R(\gamma_1, \gamma_2)$ (resp. the absolute rotation $R_{\text{abs}}(\gamma_1, \gamma_2)$) given by the Gauss integral coincides with the signed rotation $R(\gamma_2, \mathcal{L})$ (resp. the absolute rotation $R_{\text{abs}}(\gamma_2, \mathcal{L})$) of the curve γ_2 around the straight line \mathcal{L} , as defined by projection on \mathcal{L}^\perp in Section 2.1 above.*

Proof. Let us start with the case of the signed rotation. We have passing to the length parametrization

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{I_1} \int_{I_2} \frac{\langle n_1 \wedge n_2, \gamma_{12} \rangle}{\|\gamma_{12}\|^3} ds_1 ds_2,$$

where n_1, n_2 are the unit tangent vectors to the curves γ_1, γ_2 , and s_1, s_2 are the length parameters on the curves γ_1, γ_2 , respectively.

Let $\eta_2(s_2)$ denote the vector joining the point $s_2 \in \gamma_2$ with the projection of s_2 onto the straight line γ_1 (see Figure 1). We have $\langle n_1 \wedge n_2, \gamma_{12} \rangle = \langle n_1 \wedge n_2, \eta_2 \rangle$. Hence,

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{I_1} \int_{I_2} \frac{\langle n_1 \wedge n_2, \eta_2 \rangle}{\|\gamma_{12}\|^3} ds_1 ds_2.$$

The vector n_1 , being the tangent vector to the straight line γ_1 , is constant. Therefore, the triple product under the above integral depends only on s_2 , and we have

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{I_2} \langle n_1 \wedge n_2(s_2), \eta_2(s_2) \rangle \int_{I_1} \frac{ds_1}{\|\gamma_{12}(s_1, s_2)\|^3} ds_2.$$

Now the integral

$$\int_{I_1} \frac{ds_1}{\|\gamma_{12}(s_1, s_2)\|^3}$$

over the straight line γ_1 has been computed (up to a scaling by the distance $\|\eta_2(s_2)\|$ to the line) at the end of Section 2.2.1 above. It is equal to $\frac{2}{\|\eta_2(s_2)\|^2}$. So for the rotation $R(\gamma_1, \gamma_2)$, we finally get

$$R(\gamma_1, \gamma_2) = \frac{1}{2\pi} \int_{I_2} \frac{\langle n_1 \wedge n_2(s_2), \eta_2(s_2) \rangle}{\|\eta_2(s_2)\|^2} ds_2.$$

Now we can replace the vector n_2 in the triple product by its projection \hat{n}_2 to the line orthogonal to the lines γ_1 and η_2 . Hence, this triple product is equal to $\frac{\|\hat{n}_2\|}{\|\eta_2\|}$ with the sign defined by the orientation, and we obtain

$$(1) \quad R(\gamma_1, \gamma_2) = \frac{1}{2\pi} \int_{\gamma_2} \pm \frac{\|\hat{n}_2\|}{\|\eta_2\|} ds_2.$$

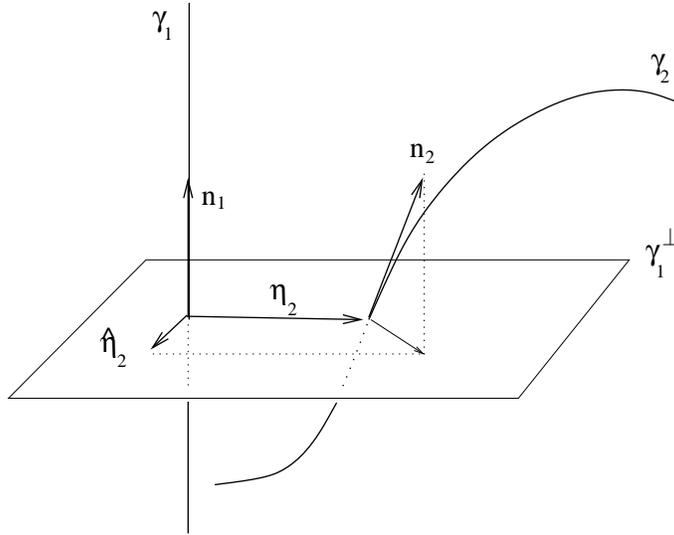


Figure 1.

But η_2 is just the radius-vector of the projection of the curve γ_2 onto the plane orthogonal to the line γ_1 , and \hat{n}_2 is the orthogonal component of the velocity vector of this projection. Therefore, according to Lemma 2.1 and up to a sign ϵ , the integral (1) is the signed rotation $R(\gamma_2, \mathcal{L})$ of γ_2 around the straight line $\mathcal{L} = \gamma_1$, as defined in Section 2.1 (the sign ϵ is $+$, when in the computation of $R(\gamma_2, \mathcal{L})$, we have oriented the plane \mathcal{L}^\perp in such a way that if (\vec{u}, \vec{v}) is an oriented orthonormal basis of \mathcal{L}^\perp , $\vec{u} \wedge \vec{v} = -n_1$ (see Figure 1)).

This completes the proof of the proposition for the signed rotation.

The proof for the absolute rotation is exactly the same: we just take the absolute value of the triple product in each step. Since the proof above consisted of a chain of point-wise equalities of the integrands, it remains valid also for the absolute values. q.e.d.

3. Rotation of trajectories of Lipschitz vector fields

For a Lipschitz vector field in \mathbb{R}^n , the very simple and basic fact is that the angular velocity of its trajectories with respect to any *stationary* point is bounded by the Lipschitz constant K . As an immediate consequence, the rotation of a trajectory around a stationary point is bounded (up to some universal constant C) by the Lipschitz constant K and the time interval T . The bound has this form: $C \cdot K \cdot T$.

Below we remind the proof of this fact. In fact, we show that the same is true for the rotation speed of the trajectories around any linear subspace $\mathcal{L} \subset \mathbb{R}^n$, which is an invariant submanifold of our field.

The main result of this section (Theorem 3.8) is that the rotation of a trajectory around any point (*stationary or not*) is bounded in terms of K and T . As the consequence, we prove (Theorem 3.9) that the mutual rotation of any two trajectories of a Lipschitz vector field is essentially bounded by the Lipschitz constant and the length of the time interval. More accurately, the absolute rotation of any two trajectories of a Lipschitz vector field on the time intervals T_1 and T_2 , respectively, is bounded from above by a linear combination of the expressions $K \cdot \min(T_1, T_2)$ and $C \cdot K^2 \cdot T_1 \cdot T_2$. Easy examples given in the end show that this bound is sharp.

At this point, let us remember that the rotation of a curve $\omega : I \rightarrow \mathbb{R}^n$ around a point, an affine space or another curve, as defined above, only depends on the trajectory $\omega(I)$, provided we only admit injective (parametrization of) curves. Consequently, our bounds do not really depend on the field, but rather on the geometry of the trajectories of the fields. This is obvious when we look at the type of bounds we obtain: (up to some universal constant) they are combinations of $K \cdot T$ and $K^2 \cdot T_1 \cdot T_2$, expressions that are unchanged with respect to any transformation of the field that preserve the trajectories.

For Lipschitz vector fields, we can summarize the situation by the following slogan: “*two trajectories have finite mutual rotation in finite time.*”

We cannot expect bounds of this sort to be true for a rotation around a non-invariant subspace of the field. Indeed, Example 3.3 below shows that a trajectory of a C^∞ -vector field in \mathbb{R}^3 can make in *finite* time an *infinite* number of turns around a straight line.

Let us remember that it was shown (in some special cases) in [Gr-Yo] that for a trajectory of a *polynomial* vector field (trajectory, which is not, in general, in some o-minimal structure), its rotation rate around any algebraic submanifold is bounded in terms of the degree of the submanifold and the degree and size of the vector field (see also [No-Ya 1, No-Ya 2, Ya]). As a consequence, we obtain also a linear in time bound on the number of intersections of the trajectory with any algebraic hypersurface.

On the other hand, our bounds on the rotation rate of the trajectories of a polynomial vector field imply upper bounds on the multiplicities of the local intersections of such trajectories with algebraic submanifolds in terms of the degree only.

As Example 3.3 shows, nothing of this sort can be expected even for C^∞ (and of course, for Lipschitz) vector fields. Still, the results of this section show that Lipschitz vector fields exhibit rather strong non-oscillation patterns. As far as the rotation of the trajectories of such fields around non-invariant submanifolds is concerned, our current understanding is far from being sufficient. In particular, there is a serious gap between the result of Theorem 3.8 below that a “global” rotation rate of a trajectory of a Lipschitz vector field around a *non-stationary point* is still bounded, and Example 3.3, demonstrating an infinite rotation around a non-invariant straight line.

3.1. Rotation of a trajectory around a stationary point. Let v be a vector field defined in a certain domain U in \mathbb{R}^n . We shall always assume v satisfying a Lipschitz condition with the constant K :

$$\|v(x) - v(y)\| \leq K\|x - y\|, \quad \text{for any two points } x, y \in U.$$

Let $x_0 \in U$ be a stationary (or a singular) point of v , i.e., $v(x_0) = 0$, so that the constant curve $c(t) = x_0, t \in \mathbb{R}$, is the integral curve of v passing through x_0 . Then, for any $x \in U$, $x \neq x_0$, the angular velocity of the trajectory of v passing through x with respect to x_0 is equal to $\|\hat{v}(x)\|/\|x - x_0\|$, where $\hat{v}(x)$ is the projection of the vector $v(x)$ to the hyperplane orthogonal to $x - x_0$. Hence, this angular velocity does not exceed K :

$$\frac{\|\hat{v}(x)\|}{\|x - x_0\|} \leq \frac{\|v(x)\|}{\|x - x_0\|} = \frac{\|v(x) - v(x_0)\|}{\|x - x_0\|} \leq K,$$

where K is the Lipschitz constant of v . By Lemma 2.1, we obtain:

Proposition 3.1. *For any trajectory $\omega(t)$ of the field v , its rotation around the stationary point x_0 of the field between the time moments t_1 and t_2 , i.e., the length of the spherical curve $s(t) = \frac{\omega(t) - x_0}{\|\omega(t) - x_0\|}$ between t_1 and t_2 , does not exceed $K \cdot (t_2 - t_1)$.*

Remark. Of course, on an infinite time interval, a trajectory may have infinite local rotation around a stationary point, as shown by the example given below (we avoid the obvious example of a cyclic trajectory, since we aim to work with injective trajectories). This example can be compared also with the result of Theorem 3.13 below.

Let us consider the following algebraic field in \mathbb{R}^2 with singular point $O = (0, 0)$:

$$v(x, y) = \left((x^2 + y^2 - 1)x - y, (x^2 + y^2 - 1)y + x \right),$$

and introduce the following notations: $r^2 = x^2 + y^2$, $\vec{\tau} = (x, y)/r$ and $\vec{n} = (-y, x)/r$. We have

$$v(x, y) = r(r^2 - 1) \cdot \vec{\tau} + r \cdot \vec{n}.$$

A trajectory passing at a point p with $r(p) = 1$ has to be the unit circle. Now consider $\omega = (\alpha, \beta)$ an integral curve of v passing through a point q with $r(q) < 1$. This trajectory cannot go outside the open unit disc and as we have $\frac{d[r^2 \circ \omega]}{dt} = 2(\alpha\alpha' + \beta\beta')$, we obtain $\frac{d[r^2 \circ \omega]}{dt} = 2(r^2 - 1)(\alpha^2 + \beta^2) < 0$. This proves that the trajectory ω has the singular point O as limit (while the unit circle has to be a limit cycle of this trajectory). Furthermore, we know that from the point q , the limit O is approached on an infinite time interval I .

On the other hand, the velocity \tilde{v} of the spherical blowing-up σ of ω is: $\frac{1}{r} \cdot r \cdot \vec{n} = \vec{n}$. We conclude that

$$R_{\text{abs}}(\omega, O) = \int_I \|\vec{n}(w(t))\| dt = \int_I 1 dt = +\infty.$$

To finish with this remark, let us recall that such an example of spiraling trajectory does not exist for v a gradient vector field of an analytic map, as proved in [Ku-Mo-Pa], [Ku-Mo]; for such a field the length of σ is finite and the limit of the secants passing through the singular point does exist (Thom's Gradient Conjecture).

Exactly in the same way, as in Proposition 3.1, we prove the following more general result about rotation of a curve around an invariant affine space \mathcal{L} . The proof actually shows that in Proposition 3.1, the assumption $v(x_0) = 0$ has preferably to be considered as " $v(x_0)$ is tangent to

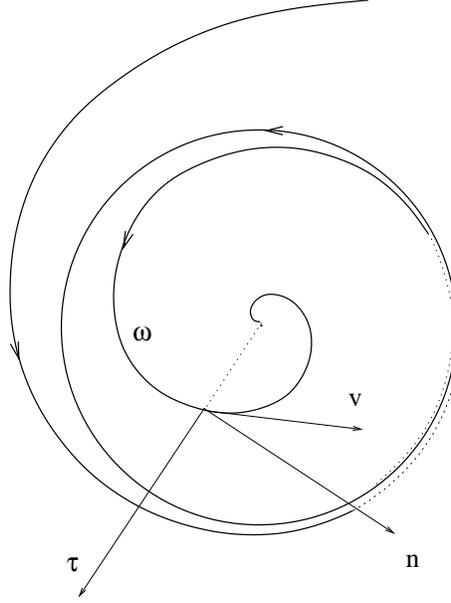


Figure 2.

the submanifold $\{x_0\}$ of \mathbb{R}^n , or “ v is a stratified field with respect to the stratification $(\{x_0\}, U \setminus \{x_0\})$ of U .”

Proposition 3.2. *For any affine subspace $\mathcal{L} \subset \mathbb{R}^n$ which is invariant for the vector field v (i.e., for any $x \in \mathcal{L}$, $v(x)$ is tangent to \mathcal{L} or, in other words, v is stratified with respect to $(\mathcal{L}, U \setminus \mathcal{L})$), the rotation speed of v in the orthogonal to \mathcal{L} direction is bounded by K . In particular, the absolute rotation of any trajectory ω of the field v around \mathcal{L} in time T does not exceed $K \cdot T$.*

Proof. According to the definition of the absolute rotation around a linear subspace (Section 2.1), we consider the orthogonal to \mathcal{L} component $\tilde{v}(x)$ of the vector field $v(x)$. Let x_0 be the projection of x onto \mathcal{L} . We denote by $\hat{v}(x)$ the “rotation” component of $\tilde{v}(x)$, orthogonal to $x - x_0$. Then the rotation speed of v in the orthogonal to \mathcal{L} direction is equal to $\|\hat{v}(x)\|/\|x - x_0\|$. Hence, this rotation speed is bounded as follows:

$$\frac{\|\hat{v}(x)\|}{\|x - x_0\|} \leq \frac{\|\tilde{v}(x)\|}{\|x - x_0\|} = \frac{\|\tilde{v}(x) - \tilde{v}(x_0)\|}{\|x - x_0\|} \leq \frac{\|v(x) - v(x_0)\|}{\|x - x_0\|} \leq K.$$

Here we use the fact that \mathcal{L} is invariant for v and hence $\tilde{v}(x_0) = 0$. By definition, we obtain that the absolute rotation of any trajectory $w(t)$ of the field v around \mathcal{L} in time T does not exceed $K \cdot T$. This completes the proof of Proposition 3.2. q.e.d.

If x_0 is not a stationary point of v , the angular velocity of $v(x)$ with respect to x_0 tends to infinity, as x approaches x_0 in any direction transverse to $v(x_0)$. Indeed, as x approaches x_0 , $v(x)$ tends to $v(x_0) \neq 0$. By Lemma 2.1, the angular velocity of $v(x)$ with respect to x_0 is equal to $\|\hat{v}(x)\|/\|x - x_0\|$, where $\hat{v}(x)$ is the component of $v(x)$ orthogonal to $x - x_0$. So if x approaches x_0 in a direction transverse to $v(x_0)$, $\hat{v}(x)$ tends to $\hat{v}_0 \neq 0$, while $x - x_0$ tends to zero (see Figure 3).

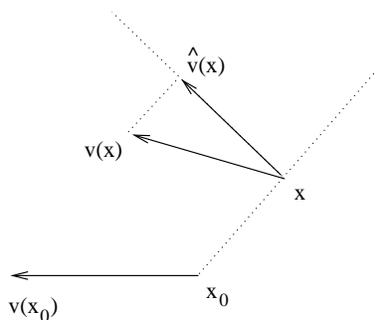


Figure 3.

However, one can show (see Theorem 3.8 below) that for any trajectory $\omega(t)$ of v , and for $t_2 - t_1$ big enough, the length of the spherical curve $s(t) = \omega(t) - x_0 / \|\omega(t) - x_0\|$ (i.e., the absolute rotation of $\omega(t)$ around x_0) is still bounded by $C \cdot K \cdot (t_2 - t_1)$. The intuitive explanation is as follows: as the trajectory $\omega(t)$ passes very close to x_0 , its rotation around x_0 in the time interval $[t_1, t_2]$ comes to approximately $1/2$ (see Figure 4).

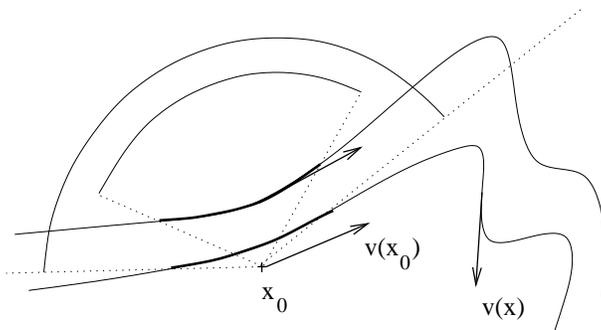


Figure 4.

However, continuing to move in the same direction, the trajectory $\omega(t)$ cannot gain more rotation. So for the total rotation of the trajectory $\omega(t)$ around x_0 to grow, its velocity vector $v(\omega(t))$ must change its direction. On the other side, if we know a priori that *the directions of*

$v(x)$ and $v(x_0)$ are strongly different with respect to the distance $\|x-x_0\|$, then the argument applied in the stationary case works, and rotation speed satisfies the same Lipschitz upper bound as above. Consequently, the time interval $[t_1, t_2]$ has to be large in order to gain a large total rotation.

Although the rotation in finite time of a trajectory of a Lipschitz field around a point is bounded, the rotation of a trajectory of a Lipschitz vector field v around a straight line, *which is not invariant under v* , can be unbounded during a finite interval of time, and this phenomena may occur even for v a C^∞ -vector field. Consider for instance the following field:

Example 3.3. Let $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a diffeomorphism, defined by

$$\Phi(x_1, x_2, x_3) = (x_1, x_2, x_3), \quad x_1 \leq 0,$$

$$\Phi(x_1, x_2, x_3) = (x_1, x_2 + \omega_1(x_1), x_3 + \omega_2(x_1)), \quad x_1 \geq 0,$$

where $\omega_1(x_1) = e^{-1/x_1^2} \cos(1/x_1)$, and $\omega_2(x_1) = e^{-1/x_1^2} \sin(1/x_1)$.

One can easily check that Φ is a C^∞ -diffeomorphism of a neighborhood of $0 \in \mathbb{R}^3$. Now the image of the positive x_1 -semiaxis under Φ is a line w , which makes an infinite number of turns around Ox_1 in any neighborhood of the origin.

Consider the vector field v in \mathbb{R}^3 , which is an image under Φ of the constant vector field $e_1 = (1, 0, 0)$. Clearly, w is a trajectory of the C^∞ -vector field v , and it makes an infinite number of turns around Ox_1 in finite time. In coordinates,

$$v(x_1, x_2, x_3) = D\Phi_{(x_1, x_2, x_3)}(e_1) = \frac{\partial \Phi}{\partial x_1}(x_1, x_2, x_3) = (1, \omega'_1(x_1), \omega'_2(x_1)).$$

Notice that in this example, the orthogonal components of v on the line Ox_1 itself have an infinite number of sign changes, accumulating to the origin.

Remark. Notice that the rotation of any two trajectories of the vector field of Example 3.3 one around another is zero. Indeed, the field v does not depend on the coordinates x_2, x_3 . So the vector joining the intersection points of the two trajectories with the planes $x_1 = c$ remains constant. In particular, this shows that we cannot expect any “transitivity” in the rotation of three curves: take two trajectories of v “far away” from the line Ox_1 , while the third trajectory is w as in Example 3.3.

3.2. Rotation of a trajectory around a non-stationary point. As it was mentioned above, one cannot bound uniformly the *momentary angular velocity* of trajectories of a vector field v with respect to a *non-stationary* point x_0 . We shall show in this section that nevertheless the

“long-time” rotation rate of trajectories of a Lipschitz field v with respect to any point x_0 , stationary or non-stationary, is uniformly bounded.

The proof of this fact given below is elementary but rather involved. Let us outline here its main idea. Assuming that the rotation of a trajectory ω of a Lipschitz field v around the point x_0 is large, we shall find a line ℓ and two points x_1, x_2 such that *the projections of the velocity vectors of ω at x_1 and x_2 on ℓ are large and point in opposite directions*. Comparing x_1 and x_2 with x_0 , we then find that the Lipschitz condition for v is violated at least at one of the points x_1, x_2 . (The detailed proof below is direct, not “by contradiction.”)

To find such two points x_1, x_2 , we first recall the proof of the classical Crofton’s formula: the length of a curve σ is equal to the average of the number of the points in its hyperplane sections. Next, using the same kind of computations, we provide more “quantitative” information on the hyperplane sections of a *parametrized* curve σ : for some of these sections at a large proportion of the intersection points the velocity vector of σ is large. Next, *assuming that the curve σ is closed*, we find hyperplane sections for which there is a couple of the intersection points with large velocity vectors, pointing in the opposite directions. Finally, we show that under the above assumption of a “large rotation” we can “close up” our curve in such a way that the points x_1, x_2 belong to the original non-closed one.

Let us fix the notations. For H a $(n - 1)$ -vector subspace of \mathbb{R}^n , we denote by λ the sphere $H \cap S^{n-1} \simeq S^{n-2}$ and by $\ell = \ell(\lambda)$ the line H^\perp . In this way, each great sphere $\lambda = H \cap S^{n-1} \simeq S^{n-2}$ is identified with the point $\ell(\lambda) \in \mathbb{R}P^{n-1}$ in the real projective space $\mathbb{R}P^{n-1}$ (we can consider instead of $\mathbb{R}P^{n-1}$ the upper half-sphere S_+^{n-1} and the points $\ell(\lambda) = \ell \cap S_+^{n-1}$).

In the rest of our constructions it may help to keep as the main example the case $n = 3$ (only this special case is used in Section 4 below): here some of the constructions become geometrically more straightforward.

For a given $x \in S^{n-1}$, consider the projective hyperplane $W(x) \subset \mathbb{R}P^{n-1}$ consisting of all the great spheres λ which contain x : $W(x) = \{\ell(\lambda) \in \mathbb{R}P^{n-1}, x \in \lambda\}$.

Now we need the following simple construction: let W_1, W_2 be two projective hyperplanes in $\mathbb{R}P^{n-1}$ and let $z_1 \in W_1, z_2 \in W_2$. Consider the rigid rotation $\rho(W_1, W_2, z_1, z_2)$ of the projective space $\mathbb{R}P^{n-1}$ which maps (W_1, z_1) to (W_2, z_2) , and which is defined as a composition of the two rotations ω_1, ω_2 . Here ω_1 rotates $\mathbb{R}P^{n-1}$ around the $(n - 3)$ -dimensional subspace $V_1 = W_1 \cap W_2$, bringing W_1 to W_2 . Then ω_2 rotates $\mathbb{R}P^{n-1}$ around the $(n - 3)$ -dimensional subspace $V_2 = (\text{span}\{z_2, \omega_1(z_1)\})^\perp$, bringing $\omega_1(z_1)$ to z_2 (see Figure 5).

Let us remark that the rigid rotation $\rho(W_1, W_2, z_1, z_2)$ depends piecewise-analytically on its arguments. In particular, for W_1, z_1 fixed, $\rho(W_1,$

$W_2, z_1, z_2)$ is analytic in W_2, z_2 everywhere but on a codimension one subset Z .

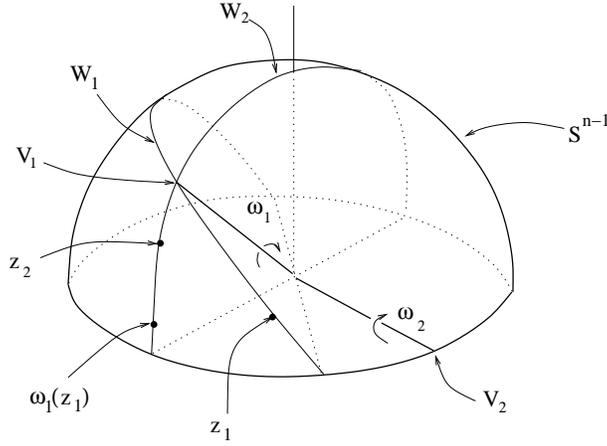


Figure 5.

Now let a unit tangent vector v to the sphere S^{n-1} at the point x be given. Then we can identify the hyperplane $W(x)$ of all the great spheres λ which contain x , with the standard projective space $W = \mathbb{R}P^{n-2}$, given in $\mathbb{R}P^{n-1}$ by $x_1 = 0$. This is done via the isometry $\rho(W, W(x), n, v)$ constructed above. Here $n = (0, 0, \dots, 0, 1)$ denotes the “north pole” of S^{n-1} , and v is the given unit tangent vector to the sphere S^{n-1} at the point x , considered as a point in $W(x)$.

Let us assume now that a differentiable curve $\sigma : [0, T] \rightarrow S^{n-1}$ is given. We denote, as above, by $v(t)$ the velocity vector of σ , $v(t) = \frac{d\sigma}{dt}(t)$, and we assume that $v(t) \neq 0$ for $t \in [0, T]$, so we define $\tilde{v}(t)$ as $\frac{v(t)}{\|v(t)\|}$. Let M denote the product $M = [0, T] \times W$. We define a mapping $\Psi : M \rightarrow \mathbb{R}P^{n-1}$ in the following way:

$$\Psi(t, z) = \rho(W, W(\sigma(t)), n, \tilde{v}(t))(z).$$

Identifying $\ell(\lambda(t, z)) = \Psi(t, z)$ with $\lambda(t, z)$, the mapping Ψ associates to the point (t, z) in M the great sphere $\lambda(t, z)$ which contains $\sigma(t)$, and which is obtained from the point $z \in W$ by application of the rotation $\rho(W, W(\sigma(t)), n, \tilde{v}(t)) : W \rightarrow W(\sigma(t))$. In particular, $\Psi(t, n)$ is the great sphere $\lambda(t, z)$ which passes through $\sigma(t)$, and which is orthogonal to $v(t)$. The mapping Ψ is at least as smooth as $v(t)$. Finally, for $z \in W$, we define $\phi(z) \in [0, \frac{\pi}{2}]$ as the (shortest) angle between the lines z and n . So $\phi(z)$ is also the angle between the normal line $\ell(\lambda(t, z)) = \Psi(t, z)$ to the great sphere $\lambda(t, z)$ and the vector $v(t)$ (see Figure 6).

Later we shall apply the above construction to curves σ for which the mapping $t \rightarrow (W(\sigma(t)), \tilde{v}(t))$ is transversal to the discontinuity set Z

of ρ (see the proof of Theorem 3.4). So all the computations below are valid but for a finite set of values of the parameter t , which does not affect the integrals.

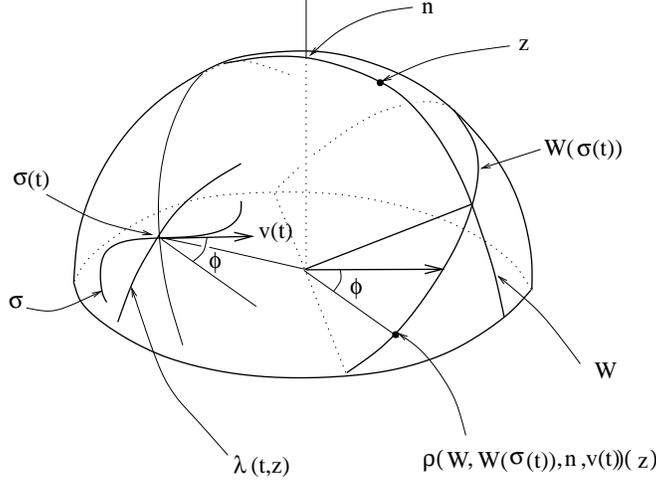


Figure 6.

We consider all the projective spaces with their standard metric and volume form (dl and dz for $\mathbb{R}P^{n-1}$ and W respectively) induced from \mathbb{R}^n . Since $\rho(W, W(\sigma(t)), n, \tilde{v}(t)) : W \rightarrow W(\sigma(t))$ is an isometry, the restriction of Ψ on $\{t\} \times W$ is an isometry for each fixed t . Now, the displacement of $\Psi(t, z)$ in the direction orthogonal to $W(\sigma(t))$, corresponding to the increment Δt , is equal (up to a higher order in Δt) to $\cos \phi(z) \|v(t)\| \Delta t$ (see Figure 6). By the classical area formula, we thus obtain

$$\int_{\lambda \in \mathbb{R}P^{n-1}} \#\{\Psi^{-1}(\lambda)\} d\ell(\lambda) = \int_{[0, T] \times W} \|v(t)\| \cos(\phi(z)) dz dt.$$

Denoting $L(\sigma)$ the total length of the curve σ , the obvious fact that $\#\{\Psi^{-1}(\lambda)\} = \#\{\sigma \cap \lambda\}$ gives

$$\begin{aligned} \int_{\lambda \in \mathbb{R}P^{n-1}} \#\{\sigma \cap \lambda\} d\ell(\lambda) &= \int_{[0, T]} \|v(t)\| dt \int_W \cos(\phi(z)) dz \\ &= L(\sigma) \int_{\phi \in [0, \frac{\pi}{2}]} \mu_{n-2} \sin^{n-3}(\phi) \cos(\phi) d\phi \\ &= \frac{\mu_{n-2}}{n-2} L(\sigma), \end{aligned}$$

where μ_r denotes the area of the unit sphere S^{n-1} .

We have proved the classical spherical Cauchy-Crofton formula (see [Fe 2, 3.2.48]):

Spherical Crofton's formula. *The total length $L(\sigma)$ of a spherical curve σ satisfies*

$$L(\sigma) = \frac{n-2}{\mu_{n-2}} \int_{\lambda \in \mathbb{R}P^{n-1}} \#\{\sigma \cap \lambda\} d\ell(\lambda) = \frac{2\pi}{\mu_n} \int_{\lambda \in \mathbb{R}P^{n-1}} \#\{\sigma \cap \lambda\} d\ell(\lambda).$$

This implies, in particular, that the average number of the intersection points of σ with the great spheres λ is $\frac{L(\sigma)}{\pi}$ (since the area of $\mathbb{R}P^{n-1}$ is $\frac{\mu_n}{2}$), and hence for some λ there is at least this number of the intersection points. Our purpose is to show that we can find λ with a large part of the intersection points of $\sigma \cap \lambda$ satisfying certain lower bounds on the angle between σ and λ and on the velocity of σ at these intersection points. Somewhat unexpectedly, these bounds resemble the quantitative transversality, or quantitative Sard theorem-type results (see [Yo-Co]).

In what follows, we always assume our curves $\sigma(t)$ to be differentiable, with a Lipschitz velocity $\frac{d\sigma}{dt}$. This assumption is satisfied for the spherical projections of the trajectories of Lipschitz vector-fields we are interested in. Notice, however, that in Theorems 3.4-3.6 and in Corollary 3.7 this assumption can be significantly relaxed: in this result only the length of the curve σ enters explicitly. We do not touch this question in the present paper.

On the other hand, we cannot assume a priori that the velocity vector $\frac{d\sigma}{dt}$ does not vanish, as required in the definition of the mapping Ψ above. Indeed, this will happen with the spherical projections of the trajectories of Lipschitz vector-fields to the sphere centered at x_0 each time as the velocity vector $v(x)$ of the vector-field points in a radial direction $x - x_0$. Also, the a priori regularity of the mapping Ψ is only Lipschitz, so an application of the above area computation would require an additional justification. To overcome these problems in the proof of Theorems 3.4-3.6 we use a smooth approximation of the curve σ .

Theorem 3.4. *Let $\sigma : [0, T] \rightarrow S^{n-1}$ be a curve of length $L(\sigma)$ and fix $\alpha, \beta \in]0, 1[$.*

- 1- *There exist $(n-2)$ -spheres $\lambda \subset S^{n-1}$ such that at least $(1 - \alpha)^{n-2}(1 - \beta)\frac{L(\sigma)}{\pi}$ points $x \in \sigma \cap \lambda$ satisfy:

 - i- *the angle between λ and $v(x)$ at x is $\geq \alpha\frac{\pi}{2}$,*
 - ii- *$\|v(x)\| \geq \beta\frac{L(\sigma)}{T}$.**
- 2- *There exist $(n-2)$ -spheres $\lambda \subset S^{n-1}$ such that the number of points in $\sigma \cap \lambda$ having properties i and ii above is $> A \cdot \#(\sigma \cap \lambda) > 0$, where $A = (1 - \alpha)^{n-2}(1 - \beta)$.*

Proof. We can assume that the curve σ is C^∞ -smooth and that its velocity does not vanish. We can assume as well that the mapping $t \rightarrow (W(\sigma(t)), \tilde{v}(t))$ is transversal to the discontinuity set Z of the rotation mapping ρ defined above. Indeed, let us approximate, up to an error ϵ , the spherical curve $\sigma(t)$ in a C^1 -topology by a C^∞ spherical curve $\sigma_\epsilon(t)$ with a non-vanishing velocity and with the required transversality condition (these properties are generic). Now assuming that Theorem 3.4 is valid for such curves, we get a $(n-2)$ -sphere $\lambda_\epsilon \subset S^{n-1}$ and the intersection points x_ϵ of λ_ϵ with σ_ϵ with the properties (i) and (ii). These properties guarantee that the intersection points x_ϵ are transversal, with the explicit lower “transversality bound” not depending on ϵ . Hence for ϵ sufficiently small we get near each point x_ϵ an intersection point of λ_ϵ with σ . As ϵ tends to zero, we get the required result for the original curve σ . To simplify the presentation, we shall not repeat this construction in the proofs of Theorems 3.5 and 3.6.

We adopt the following notations:

$$I_\beta = \left\{ t \in [0, T]; \|v(t)\| \geq \beta \frac{L(\sigma)}{T} \right\},$$

$$W_\alpha = \left\{ z \in W; \phi(z) \geq \alpha \frac{\pi}{2} \right\} \quad \text{and} \quad M_{\alpha, \beta} = I_\beta \times W_\alpha.$$

We then have $\int_{t \in I_\beta} \|v(t)\| dt \geq (1 - \beta)L(\sigma)$ since $\int_{t \in [0, T] \setminus I_\beta} \|v(t)\| dt \leq T\beta \frac{L(\sigma)}{T}$ and on the other hand,

$$\begin{aligned} \int_{z \in W_\alpha} \cos(\phi(z)) dz &= \mu_{n-2} \int_{[0, (1-\alpha)\frac{\pi}{2}]} \sin^{n-3}(\phi(z)) \cos(\phi(z)) dz \\ &= \frac{\mu_{n-2}}{n-2} \sin^{n-2}\left((1-\alpha)\frac{\pi}{2}\right) \\ &\geq \frac{\mu_{n-2}}{n-2} (1-\alpha)^{n-2}, \end{aligned}$$

by the the convexity of \sin on $\left[0, \frac{\pi}{2}\right]$. We obtain

$$\begin{aligned} &\int_{(t,z) \in M_{\alpha, \beta}} \|v(t)\| \cdot \cos(\phi(z)) dt dz \\ &= \int_{t \in I_\beta} \|v(t)\| dt \cdot \int_{z \in W_\alpha} \cos(\phi(z)) dz \\ &\geq (1 - \beta)L(\sigma) \cdot \frac{\mu_{n-2}}{n-2} (1 - \alpha)^{n-2} \\ &= \frac{\mu_n}{2\pi} (1 - \beta)(1 - \alpha)^{n-2} L(\sigma). \end{aligned}$$

Finally, for $\lambda \in \mathbb{R}P^{n-1}$, $\#\{\Psi^{-1}(\lambda) \cap M_{\alpha,\beta}\} = \#\{x \in \sigma \cap \lambda; x \text{ satisfies i and ii}\}$; therefore, by the area formula again:

$$\begin{aligned}
 (**) \quad \int_{\lambda \in \mathbb{R}P^{n-1}} \#\{x \in \sigma \cap \lambda; x \text{ satisfies i and ii}\} d\ell \\
 \geq \frac{\mu_n}{2\pi} (1-\beta)(1-\alpha)^{n-2} L(\sigma).
 \end{aligned}$$

Let us now prove the first statement. By inequality (**), there necessarily exists $\lambda \subset S^{n-1}$ such that:

$$\begin{aligned}
 \text{Vol}(\mathbb{R}P^{n-1}) \cdot \#\{x \in \sigma \cap \lambda; x \text{ satisfies i and ii}\} \\
 \geq \frac{\mu_n}{2\pi} (1-\beta)(1-\alpha)^{n-2} L(\sigma),
 \end{aligned}$$

and $\text{Vol}(\mathbb{R}P^{n-1}) = \frac{\mu_n}{2}$.

We prove the second statement in the same way. If for all $\lambda \subset S^{n-1}$, the number of points in $\sigma \cap \lambda$ satisfying properties i and ii is $< A \cdot \#(\sigma \cap \lambda)$, by the spherical Crofton formula and then by inequality (**) we have

$$\begin{aligned}
 A \cdot \frac{\mu_n}{2\pi} L(\sigma) &= A \int_{\lambda \in \mathbb{R}P^{n-1}} \#(\sigma \cap \lambda(\ell)) d\ell \\
 &> \int_{\lambda \in \mathbb{R}P^{n-1}} \#\{x \in \sigma \cap \lambda(\ell); x \text{ satisfies i and ii}\} d\ell \\
 &\geq \frac{\mu_n}{2\pi} (1-\beta)(1-\alpha)^{n-2} L(\sigma),
 \end{aligned}$$

which is a contradiction, since $A = (1-\beta)(1-\alpha)^{n-2}$. q.e.d.

Theorem 3.5. *Let $\sigma : [0, T] \rightarrow S^{n-1}$ be a closed curve of length $L(\sigma)$. For all $\alpha \in]0, 1 - (\frac{1}{2})^{\frac{1}{n-2}}[$, there exist a $(n-2)$ -sphere $\lambda \subset S^{n-1}$ and two points $x_1, x_2 \in \sigma \cap \lambda$ such that:*

- i- *The angle between λ and $v(x_j)$ at x_j is $\geq \alpha \frac{\pi}{2}$, for $j \in \{1, 2\}$,*
- ii- *For $j \in \{1, 2\}$, $\|v(x_j)\| \geq (1 - \frac{1}{2(1-\alpha)^{n-2}}) \frac{L(\sigma)}{T}$,*
- iii- *The projections of $v(x_1)$ and $v(x_2)$ on $\ell(\lambda)$ have opposite directions.*

Proof. Taking $\beta = 1 - \frac{1}{2(1-\alpha)^{n-2}}$, we have $\beta \in]0, 1[$ for any $\alpha \in]0, 1 - (\frac{1}{2})^{\frac{1}{n-2}}[$ and for such α, β , $(1-\alpha)^{n-2}(1-\beta) = \frac{1}{2}$. By the second part of Theorem 3.4, there exists a $(n-2)$ -sphere $\lambda \subset S^{n-1}$ such that $\sigma \cap \lambda \neq \emptyset$ and for more than half of the number of points x of $\sigma \cap \lambda$, the angle between λ and $v(x)$ is at least $\alpha \frac{\pi}{2}$ and $\|v(x)\| \geq \beta \frac{L(\sigma)}{T}$. As σ is closed, the proportion of intersection points x at which σ and λ are transverse and such that $v(x)$ has a given direction along ℓ may not be

more than one half. Consequently, at least a pair of points x_1 and x_2 have the three properties required. q.e.d.

We extend Theorem 3.5 above to non-closed spherical curves.

Theorem 3.6. *Let $\sigma : [0, T] \rightarrow S^{n-1}$ be a curve of length $L(\sigma)$ and for all $\alpha \in]0, 1 - (\frac{1}{2})^{\frac{1}{n-2}}[$, denote $\beta = 1 - \frac{1}{2(1-\alpha)^{n-2}}$ and assume that $L(\sigma) > 2\pi\theta$, with $\theta > \frac{1}{\beta}$. There exist a $(n-2)$ -sphere $\lambda \subset S^{n-1}$ and two points $x_1, x_2 \in \sigma \cap \lambda$ such that:*

- i- *The angle between λ and $v(x_j)$ at x_j is $\geq \alpha \frac{\pi}{2}$, for $j \in \{1, 2\}$.*
- ii- *For $j \in \{1, 2\}$, $\|v(x_j)\| \geq (\beta - \frac{1}{\theta}) \frac{L(\sigma)}{T}$.*
- iii- *The projections of $v(x_1)$ and $v(x_2)$ on $\ell(\lambda)$ have opposite directions.*

Proof. In case σ is not closed, consider a new closed curve $\tilde{\sigma} : [0, \tilde{T}] \rightarrow S^{n-1}$ obtain by closing σ with some geodesic (of length $\leq 2\pi$) passed with velocity $\frac{2\pi}{\gamma \cdot T}$, for $\gamma = \frac{1}{\beta\theta - 1}$. We have $\tilde{T} \leq T + \frac{2\pi}{2\pi/(\gamma \cdot T)} = T(1 + \gamma)$. We apply now Theorem 3.5 to the closed curve $\tilde{\sigma}$: there exist a $(n-2)$ -sphere $\lambda \subset S^{n-1}$ and two points $x_1, x_2 \in \sigma \cap \lambda$ such that i and iii of Theorem 3.5 are satisfied and such that $\|v(x_j)\| \geq \beta \frac{L(\tilde{\sigma})}{\tilde{T}} \geq \beta \frac{L(\sigma)}{(1+\gamma)T} = (\beta - \frac{1}{\theta}) \frac{L(\sigma)}{T}$. On the other hand, as $\|v(x_j)\| \geq (\beta - \frac{1}{\theta}) \frac{L(\sigma)}{T} > (\beta - \frac{1}{\theta}) \frac{2\pi\theta}{T} = \frac{2\pi}{\gamma \cdot T}$, the points x_1 and x_2 cannot belong to the geodesic added to σ in our closing process. q.e.d.

Corollary 3.7. *Denoting $\nu = 1 - \frac{1}{\sqrt{2}}$, let $\sigma : [0, T] \rightarrow S^{n-1}$ be a curve of length $L(\sigma) > 2\pi\theta$, with $\theta > \frac{1}{\nu}$. There exist a line $\ell \in \mathbb{R}^n$ and two points $x_1, x_2 \in \sigma \cap \lambda(\ell)$ such that for $j \in \{1, 2\}$, the projection p_j of $v(x_j)$ on ℓ have opposite directions and $\|p_j\| \geq \frac{1}{n-2}(\nu^2 - \frac{\nu}{\theta}) \frac{L(\sigma)}{T}$.*

Proof. By Theorem 3.6, there exist a line $\ell \in \mathbb{R}^n$ and two points $x_1, x_2 \in \sigma$ such that for $j \in \{1, 2\}$, the projection p_j of $v(x_j)$ on ℓ have opposite directions and $\|p_j\| \geq \sin(\alpha \frac{\pi}{2})(1 - \frac{1}{2(1-\alpha)^{n-2}} - \frac{1}{\theta}) \frac{L(\sigma)}{T}$, for all $\alpha \in]0, 1 - (\frac{1}{2})^{\frac{1}{n-2}}[$ and for $\frac{1}{\theta} < 1 - \frac{1}{2(1-\alpha)^{n-2}}$. Now, for such values of α , we have $\sin(\alpha \frac{\pi}{2}) \geq \alpha$. Furthermore, for all $\theta > 2$ and for

$\alpha \in]0, \frac{1}{2(n-2)} \frac{\theta-2}{\theta-1}[$, we have

$$1 - \frac{1}{2(1-\alpha)^{n-2}} - \frac{1}{\theta} \geq 1 - \frac{1}{2(1-(n-2)\alpha)} - \frac{1}{\theta} > 0.$$

Denoting $h_\theta = \sup_{\alpha \in]0, \frac{1}{2(n-2)} \frac{\theta-2}{\theta-1}[} \alpha(1 - \frac{1}{2(1-(n-2)\alpha)} - \frac{1}{\theta})$, we obtain

$$\|p_j\| \geq h_\theta \cdot \frac{L(\sigma)}{T}, \quad \text{for any } \theta > 2.$$

A direct computation shows that h_θ is reached at $\alpha = \frac{1}{n-2}(1 - \sqrt{\frac{\theta}{2(\theta-1)}})$ and that $h_\theta = \frac{1}{n-2}(\sqrt{1 - \frac{1}{\theta}} - \frac{1}{\sqrt{2}})^2 > \frac{1}{n-2}(\nu^2 - \frac{\nu}{\theta})$ (note that $c = \nu^2$ and $d = \nu$ are the sharpest values of c and d such that $h_\theta > c + d/\theta$, and that $\nu^2 - \frac{\nu}{\theta} > 0$ for $\theta > 1/\nu > 2$). q.e.d.

Our main result is the following theorem:

Theorem 3.8. *Let v be a Lipschitz vector field on an open set $U \subset \mathbb{R}^n$ with Lipschitz constant K and let $\omega(t)$ be a trajectory of v . Then for any $x_0 \in U$ the absolute rotation of ω around x_0 between any two time moments t_1 and t_2 satisfies*

$$R_{\text{abs}}(\omega, x_0) \leq \frac{2\pi}{\nu} + \frac{(n-2)K}{\nu^2} \cdot (t_2 - t_1),$$

with $\nu = 1 - \frac{1}{\sqrt{2}}$.

Proof. We apply Corollary 3.7 to the spherical curve $\sigma(t) = (\omega(t) - x_0)/\|\omega(t) - x_0\|$ between the moments t_1 and t_2 . By this corollary, for any $\theta > 1/\nu$, either the length $s(t_1, t_2)$ of $\sigma(t)$ between the moments t_1 and t_2 is smaller than $2\pi\theta$, or there exist a line ℓ and two time parameters τ_1, τ_2 such that for $j \in \{1, 2\}$, $\sigma(\tau_j) \in \lambda(\ell)$, the projections p_j on ℓ of the velocity vectors of σ at τ_j have opposite directions and $\|p_j\| \geq \frac{1}{n-2}(\nu^2 - \frac{\nu}{\theta}) \frac{s(t_1, t_2)}{t_2 - t_1}$. We adopt the following notations: $v(t)$ is the velocity vector of the trajectory $\omega(t)$, while $\tilde{v}(t)$ is the velocity vector of the spherical curve σ . Then for $j \in \{1, 2\}$, $\omega_j = \omega(\tau_j)$, $v_j = v(\tau_j)$, $\tilde{v}_j = \tilde{v}(\tau_j)$ and for a point $s \in S^{n-1}$, $T(s)$ is the tangent hyperplane to S^{n-1} at s . We remark that $\ell \subset T(s)$, for all $s \in \lambda(\ell)$. The immediate consequence of this inclusion is that for $j \in \{1, 2\}$, p_j is also the projection on ℓ of the vector $\frac{v_j}{\|\omega_j - x_0\|}$, because $\tilde{v}(t)$ is the orthogonal to $\omega(t) - x_0$ component of $\frac{v(t)}{\|\omega(t) - x_0\|}$. Consequently, the projections on ℓ

of $\frac{v_j}{\|w_j - x_0\|}$ have length $\geq \frac{1}{n-2}(\nu^2 - \frac{\nu}{\theta})\frac{s(t_1, t_2)}{t_2 - t_1}$ and their directions are opposite.

Let us now consider two auxiliary vectors: $V_1 = \frac{v_1 - v(x_0)}{\|w_1 - x_0\|}$ and $V_2 = \frac{v_2 - v(x_0)}{\|w_2 - x_0\|}$. Since the projections of $\frac{v_1}{\|w_1 - x_0\|}$ and $\frac{v_2}{\|w_2 - x_0\|}$ on ℓ had opposite directions, independently of the vector $v(x_0)$ at least one of the vectors V_1, V_2 has its projection on ℓ of size $\geq \frac{1}{n-2}(\nu^2 - \frac{\nu}{\theta})\frac{s(t_1, t_2)}{t_2 - t_1}$. But since the vector field v is Lipschitz, the norms of both the vectors V_1 and V_2 are bounded by K . Hence $\frac{1}{n-2}(\nu^2 - \frac{\nu}{\theta})\frac{s(t_2, t_1)}{t_2 - t_1} \leq K$, or $s(t_2, t_1) \leq \frac{\theta}{\nu^2\theta - \nu}(n-2)K(t_2 - t_1)$. In any case we have proved

$$\begin{aligned} R_{\text{abs}}(\omega, x_0) &= s(t_2, t_1) \\ &\leq \max \left[2\pi\theta, \frac{\theta}{\nu^2\theta - \nu}(n-2)K(t_2 - t_1) \right], \quad \forall \theta > 1/\nu. \end{aligned}$$

Finally

$$\begin{aligned} R_{\text{abs}}(\omega, x_0) &\leq \inf_{\theta > \frac{1}{\nu}} \max \left(2\pi\theta, \frac{\theta}{\nu^2\theta - \nu}(n-2)K(t_2 - t_1) \right) \\ &= \frac{2\pi}{\nu} + \frac{(n-2)K}{\nu^2}(t_2 - t_1). \end{aligned}$$

q.e.d.

3.3. Rotation of two Lipschitz trajectories. In this section we prove that the absolute rotation of any two trajectories of a Lipschitz vector field in \mathbb{R}^3 is bounded in terms of the Lipschitz constant K and the time interval.

Theorem 3.9. *Let as above $\nu = 1 - \frac{1}{\sqrt{2}}$ and let ω_1 and ω_2 be trajectories, on time intervals T_1 and T_2 respectively, of a Lipschitz vector field v defined in some open subset U of \mathbb{R}^3 and of a Lipschitz constant of K . Then the mutual absolute rotation of ω_1 and ω_2 satisfies*

$$R_{\text{abs}}(\omega_1, \omega_2) \leq \frac{K}{2\nu} \min\{T_1, T_2\} + \frac{K^2}{4\pi\nu^2} T_1 T_2.$$

In fact, we shall prove a more accurate version of this theorem, which bounds the absolute rotation $R_{\text{abs}}(\omega_1, \omega_2)$ through the Lipschitz constant of the field and through the bound of the absolute rotation of the trajectory ω_1 (respectively ω_2) around the *points* of ω_2 (respectively of ω_1) (Theorem 3.10). Then to get back to Theorem 3.9 we use the uniform bound on the rotation of Lipschitz trajectories around points, provided by Theorem 3.8.

Theorem 3.10. *Let ω_1 and ω_2 be trajectories, on time intervals T_1 and T_2 respectively, of a Lipschitz vector field v defined in some open subset U of \mathbb{R}^3 . With the following notations:*

$$R_1 = \max_{p_2 \in \omega_2} R_{\text{abs}}(\omega_1, p_2), \quad R_2 = \max_{p_1 \in \omega_1} R_{\text{abs}}(\omega_2, p_1),$$

the absolute rotation of ω_1 and ω_2 satisfies

$$R_{\text{abs}}(\omega_1, \omega_2) \leq \frac{1}{4\pi} K R_1 T_2 \quad \text{and} \quad R_{\text{abs}}(\omega_1, \omega_2) \leq \frac{1}{4\pi} K R_2 T_1.$$

Corollary 3.11. *With the same notations and hypothesis as in Theorem 3.10, we have*

$$R_{\text{abs}}(\omega_1, \omega_2) \leq \frac{K}{4\pi} \cdot \min\{R_1 T_2, R_2 T_1\}.$$

Proof of Theorem 3.10. We recall that for trajectories of the vector field v the Gauss integral takes the form

$$R_{\text{abs}}(\omega_1, \omega_2) = \frac{1}{4\pi} \int_{T_1} \int_{T_2} \frac{|\langle v_1 \wedge v_2, r_{12} \rangle|}{\|r_{12}\|^3} dt_1 dt_2.$$

Here v_1, v_2 are the velocity vectors of ω_1, ω_2 , respectively (i.e., the values of the vector field v at the running points $p_1(t_1)$ and $p_2(t_2)$ on ω_1 and ω_2), and $r_{12} = p_2 - p_1$ is the vector joining the running points p_1 and p_2 . So we have

$$\begin{aligned} & 4\pi \cdot R_{\text{abs}}(\omega_1, \omega_2) \\ &= \int_{T_1} \int_{T_2} \frac{|\langle v_1 \wedge v_2, r_{12} \rangle|}{\|r_{12}\|^3} dt_1 dt_2 \\ &= \int_{t_2 \in T_2} dt_2 \int_{t_1 \in T_1} \frac{|\langle v_1(t_1) \wedge (v_2(t_2) - v_1(t_1)), r_{12}(t_1, t_2) \rangle|}{\|r_{12}\|^3} dt_1. \end{aligned}$$

Indeed, the subtraction of $v_1(t_1)$ from $v_2(t_2)$ does not change the triple product under the integral. Now, since the vector field v is Lipschitz, we have: $\|v_2(t_2) - v_1(t_1)\| \leq K \cdot \|r_{12}\|$.

Hence, for the triple product we obtain

$$|\langle v_1 \wedge (v_2 - v_1), r_{12} \rangle| \leq \|\tilde{v}_1\| \cdot K \cdot \|r_{12}\| \cdot \|r_{12}\|,$$

where \tilde{v}_1 is the component of the vector v_1 orthogonal to the vector r_{12} (see Figure 7).

Therefore for the absolute rotation we get

$$4\pi \cdot R_{\text{abs}}(\omega_1, \omega_2) \leq K \cdot \int_{t_2 \in T_2} dt_2 \int_{t_1 \in T_1} \frac{\|\tilde{v}_1(t_1, t_2)\|}{\|r_{12}(t_1, t_2)\|} dt_1.$$

In the interior integral the time t_2 and the corresponding point $p_2(t_2)$ on the curve ω_2 are fixed, while the integration runs over the curve ω_1 , and hence the integral $\int_{t_1 \in T_1} \frac{\|\tilde{v}_1(t_1, t_2)\|}{\|r_{12}(t_1, t_2)\|} dt_1$ is equal to the length of the spherical projection σ_1 of the curve ω_1 from the point $p_2(t_2)$, i.e., to

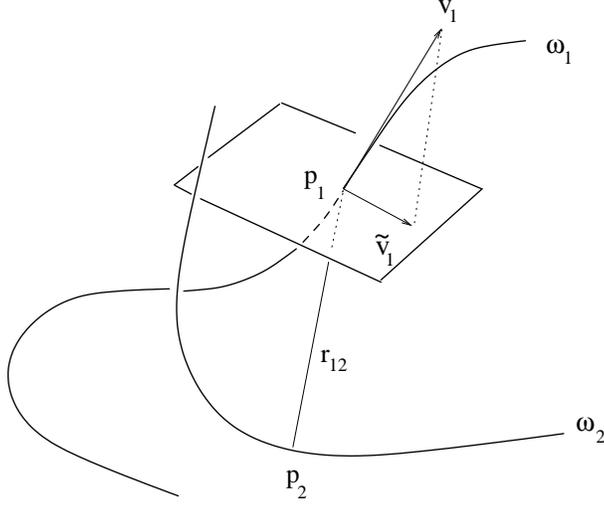


Figure 7.

$2\pi \cdot R_{\text{abs}}(\omega_1, p_2)$. By the assumptions, this rotation is uniformly in p_2 bounded by R_1 . Hence the interior integral $\int_{t_1 \in T_1} \frac{\|\tilde{v}_1\|}{\|r_{12}\|} dt_1$ does not exceed R_1 , and finally

$$4\pi R_{\text{abs}}(\omega_1, \omega_2) \leq KT_2 R_1.$$

The setting of Theorem 3.10 is symmetric with respect to the trajectories ω_1 and ω_2 . So interchanging these trajectories we get:

$$4\pi R_{\text{abs}}(\omega_1, \omega_2) \leq KT_1 R_2.$$

This completes the proof of Theorem 3.10. q.e.d.

Proof of Theorem 3.9. We use Theorem 3.8, which states that a rotation in time T of any trajectory of a Lipschitz vector field v in \mathbb{R}^3 around any point p does not exceed $\frac{2\pi}{\nu} + \frac{KT}{\nu^2}$. Applying Corollary 3.11 we get

$$R_{\text{abs}}(\omega_1, \omega_2) \leq \frac{K}{2} \min \left(\frac{T_1}{\nu} + \frac{K}{2\pi\nu^2} T_1 T_2, \frac{T_2}{\nu} + \frac{K}{2\pi\nu^2} T_1 T_2 \right),$$

and Theorem 3.9 is proved. q.e.d.

Examples of application: Asymptotic Hopf invariant and Logarithmic bound of the local rotation at singular points for analytic vector fields.

In [Ar] the following extremal problem is studied: a divergence-free vector field v being given in some compact domain U of \mathbb{R}^3 (assuming that v is tangent to the boundary of U), find the minimum energy

$e(v)$ for fields ξ obtained from v under the action of volume preserving diffeomorphisms of U (we recall that the energy $E(\xi)$ of ξ is $\frac{1}{2} \int_U (\xi, \xi)$).

The following construction is then done: for $T_1, T_2 > 0$ and for (almost all) x and y in U , let us denote γ_x the trajectory of v passing through x at $t = 0$ and Γ_{x, T_1} the closed curve obtained by closing $\gamma_x([0, T_1])$ by a segment. Then consider $R(\Gamma_{x, T_1}, \Gamma_{y, T_2})$, the rotation of Γ_{x, T_1} and Γ_{y, T_2} . It is proved that the limit $\mathcal{A}_v(x, y) = \lim_{T_1, T_2 \rightarrow \infty} \frac{1}{T_1 T_2} R(\Gamma_{x, T_1}, \Gamma_{y, T_2})$ exists and equals $\lim_{T_1, T_2 \rightarrow \infty} \frac{1}{T_1 T_2} R(\gamma_x(T_1), \gamma_y(T_2))$, where $R(\gamma_x(T_1), \gamma_y(T_2))$ is the signed rotation of the non-closed curves, as defined in Section 2.2.2. The asymptotic Hopf invariant $I(v)$ of v is then defined as $I(v) \cdot \int_{(x, y) \in U \times U} dx dy = \int_{(x, y) \in U \times U} \mathcal{A}_v(x, y) dx dy$ and it is proved (under some additional conditions) that $e(v) = I(v)/\lambda$ (where λ is an eigenvalue of a certain naturally arising operator, see [Ar], Section 5.1).

The question of whether I is, to a certain extent, a topological invariant has been studied in particular in [Ga-Gh 1] and [Ga-Gh 2].

As a direct consequence of Theorem 3.9 we obtain an upper bound for $I(v)$ in term of the Lipschitz constant K of v .

Theorem 3.12. *Let v be a divergence-free vector field in some compact domain U of \mathbb{R}^3 , tangent to the boundary of U and of Lipschitz constant K . Then the asymptotic Hopf invariant $I(v)$ of v satisfies the following inequality:*

$$I(v) \leq \frac{K^2}{4\pi\nu^2 \text{Vol}^2(U)},$$

where $\nu = 1 - \frac{1}{\sqrt{2}}$.

Proof. With the notations introduced above and from Theorem 3.9. we have

$$\frac{1}{T_1 T_2} R(\gamma_x(T_1), \gamma_y(T_2)) \leq \frac{K \min\{T_1, T_2\}}{2\nu T_1 T_2} + \frac{K^2}{4\pi\nu^2}.$$

Letting T_1 and T_2 go to ∞ , we obtain the desired inequality. q.e.d.

As another application of the bounds found in Section 3, let us now consider the following situation. The vector field

$$\frac{dv}{dt} = Lv + G(v)$$

is defined in a neighborhood of $O \in \mathbb{R}^3$ and has a non-degenerate linear part L with all the eigenvalues ℓ_j , $j = 1, 2, 3$, having a negative real part: $\Re(\ell_j) \leq \ell < 0$, $j = 1, 2, 3$. In dynamical language, $v =$ has a non-degenerate sink at the origin (it is the case of our field in \mathbb{R}^2 , in the

Remark following Proposition 3.1). It is easy to see that the Lipschitz constant of v in a neighborhood U of the origin tends to the norm of L as U shrinks to the origin. The following theorem is an immediate corollary of the results of Section 3:

Theorem 3.13. *For any two trajectories ω_1, ω_2 of the field v in a neighborhood of $O \in \mathbb{R}^3$, the absolute rotation $R_{\text{abs}}(\omega_1, \omega_2)$ grows at most logarithmically with the distance to the origin. More accurately, the rotation $R_{\text{abs}}(\omega_1, \omega_2, R, r)$ of the parts of ω_1, ω_2 between the spheres of the radii $R > r > 0$ satisfies*

$$R_{\text{abs}}(\omega_1, \omega_2, R, r) \leq C \|L\|^2 \frac{\log^2(R/r)}{\ell}.$$

Proof. This is a direct consequence of Theorem 3.9, since the Lipschitz constant of v in a neighborhood U of the origin tends to $\|L\|$ as U shrinks to the origin, while the time interval for both the trajectories between the spheres of the radii $R > r > 0$ is of order $\frac{\log(R/r)}{\ell}$. q.e.d.

Remark. Of course, one can easily show that the bound of Theorem 3.13 is sharp: consider a linear vector field

$$\frac{dv}{dt} = Lv$$

in a neighborhood of $O \in \mathbb{R}^3$, with a non-degenerate linear part L having all its eigenvalues ℓ_j , $j = 1, 2, 3$, with negative real part: $\text{Re}(\ell_j) \leq \ell < 0$, $j = 1, 2, 3$. Assume in addition that $\ell_1 \in \mathbb{R}$, while ℓ_2 and ℓ_3 are conjugate: $\ell_{2,3} = \alpha \pm i\beta$. Then the solutions are $x_1 = C_1 \cdot \exp(\ell_1 t)$, $x_2 = C_2 \cdot \exp(\alpha t) \cdot \sin(\beta t)$, $x_3 = C_3 \cdot \exp(\alpha t) \cdot \cos(\beta t)$, and the trajectories rotate around the x_1 -axis and one around another exactly as prescribed by the upper bound given in Theorem 3.9.

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