REAL SINGULARITIES & LIPSCHITZ-KILLING CURVATURES

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Notations

- $X \ni 0$ a compact subanalytic subset of R^n (or a definable set in some o-minimal structure expanding the reals),
- X is stratified by some **regular** stratification $(X^j)_{j\in\{0,\cdots,k\}}$:
  $X = \bigcup_{disjoint} X^j$, X^j smooth,
- X_0 the germ of X at 0,
- $G(i,n)$ the Grassmannian of i-dimensional vector subspaces of R^n,
- $\bar{G}(i,n)$ the Grassmannian of i-dimensional affine subspaces of R^n,
- π_P the orthogonal projection onto P for $P \in G(i,n)$

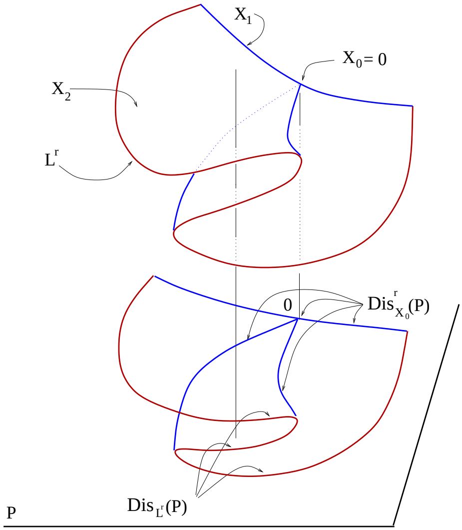
Definitions before the picture

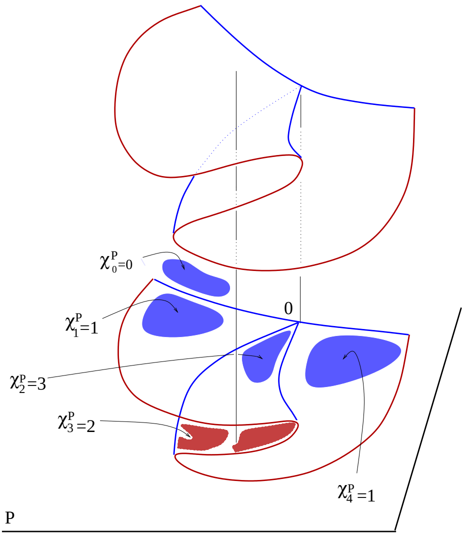
The blue objects are closed to 0

- The polar variety of X^j relative to $P \in G(i,n)$: the closure of the critical points of $\pi_{P|X^j}$
- The discriminant of the germ X_0 relative to P : the union over the X^j's of the images $\pi_P(Pol^P_{X^j \cap B(0,r)})$, $0 < r << 1$

The red objects are far from 0

- The r-link of X^j, $L^r_{X^j} := X^j \cap S(0,r)$
- The union over the X^j's of the discriminants of the r-link $L^r_{X^j}$ relative to P


X1
X0= 0
X2
Lr
0
Disr X0(P)
DisLr(P)
P


$\chi_0^P=0$
$\chi_1^P=1$
0
$\chi_2^P=3$
$\chi_3^P=2$
$\chi_4^P=1$
P

Density

The i-density $\Theta_i(K_0)$ of an i-dimensional subanalytic germ $K_0 \subset R_0^n$

$$\Theta_i(K_0) = \lim_{\epsilon \to 0} \frac{Vol_i(K \cap B(0,\epsilon))}{Vol_i(B^i(0,\epsilon))}$$

The polar invariants $\sigma_i(X_0)$

- $$\sigma_i(X_0) = \int_{P \in G(i,n)} \sum_{j=1}^{n_P} \chi_j^P \cdot \Theta_i(K_j^P) \ \ dP$$
- $\sigma_*(X_0) = (\sigma_0(X_0), \cdots, \sigma_n(X_0))$

Remarks

- The red domains are not considered in this computation, only the blue ones are.
- When $d = \dim(X_0)$ and d_0 is the dimension of the stratum containing 0 of a (b)-regular statification, one has :

$$\sigma_*(X_0) = (1, \cdots, 1, \sigma_{d_0+1}(X_0), \cdots, \sigma_d(X_0) = \Theta_d(X_0), 0 \cdots, 0).$$

Remarks in the complex case

- If X_0 is an isolated complex hypersurface singularity $\sigma_i(X_0) = 1 + (-1)^{n-i-1}\mu_{n-i}$, where μ_{n-i} is the Milnor number of the $(n-i)$-section of X_0.
- In the general complex case σ_* is a linear combination of the "evanescent characteristics" of Kashiwara which in turn are lin. comb. of the multiplicities $e(Pol^k(X_0), 0)$ of the polar varieties

Theorem Equising$_C$

(Briançon-Speder 1976, Henry-Merle 1981, Navarro 1981, Teissier 1981, Lê-Teissier 1981)
Let X_0 be a complex analytic germ at 0 in C^n stratified by $(X^j)_{j\in\{0,\cdots,k\}}$.
The following propositions are equivalent :

- $(X^j)_{j\in\{0,\cdots,k\}}$ is (b)-regular.
- $(X^j)_{j\in\{0,\cdots,k\}}$ is (w)-regular.
- The functions $y \mapsto e(Pol^k(X_y), y)$ are constant along the X^j's.
- The functions $y \mapsto e(Dis^k(X_y), y)$ are constant along the X^j's.
- The functions $y \mapsto \sigma_i(X_y)$ are constant along the X^j's.

Theorem Equising$_R$, GC-MM

- σ_* is continuous along the strata of a (w)-regular stratification
- No equivalence in the real case.

Recall that

- The i^{th}-Lipschitz-Killing invariant of X is by definition the valuation

$$\Lambda_i(X) = \int_{\bar{P}\in\bar{G}(n-i,n)} \chi(X \cap \bar{P}) \ \ \frac{d\bar{P}}{\beta(n,i)},$$

- $\Lambda_*(X) = (\chi(X), \cdots, Vol_d(X), 0, \cdots, 0)$.

Theorem

- The limit

$$\lim_{r\to 0} \frac{1}{\alpha_i . r^i} \Lambda_i(X \cap B^n_{(0,r)}) := \Lambda_i^{\ell oc}(X_0)$$

exists for $i = 0, \cdots, n$. One call it the local Lipschitz-Killing invariants of X_0.

- $\Lambda_*^{\ell oc}(X_0) = (\Lambda_1^{\ell oc}(X_0), \cdots, \Lambda_n^{\ell oc}(X_0))$

Remark

The blue and the red domains are all together considered in the computation of $\Lambda_i(X \cap B(0,r))$, so they do matter together in the defintion of $\Lambda_*^{\ell oc}$.

Remark

When X_0 is a convex cone, $X = R_+ \cdot K$, with $K \in S^{n-1}$, the functions $K \mapsto \Lambda_i^{\ell oc}(X_0)$ and $K \mapsto \sigma_i(X_0)$ are $O_n(R)$-invariant valuations on S^{n-1}. Then $(\sigma_i, \cdots, \sigma_n)$ and $(\Lambda_0^{\ell oc}, \cdots, \Lambda_n^{\ell oc})$ are basis of the space of $O_n(R)$-invariant valuations on S^{n-1}.

Theorem (Local multidimensional Crofton formula), GC-MM

In the general case (X_0 not necessarily a cone) :

$$\begin{pmatrix} \Lambda_1^{\ell oc} \\ \vdots \\ \Lambda_n^{\ell oc} \end{pmatrix} (X_0) = \begin{pmatrix} 1 & m_1^2 & \dots & m_1^{n-1} & m_1^n \\ 0 & 1 & \dots & m_2^{n-1} & m_2^n \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} (X_0)$$

with : $m_i^j = \frac{\alpha_j}{\alpha_{j-i} \cdot \alpha_i} \begin{pmatrix} i \\ j \end{pmatrix} - \frac{\alpha_{j-1}}{\alpha_{j-1-i} \cdot \alpha_i} \begin{pmatrix} i \\ j-1 \end{pmatrix}$, if $i+1 \leq j \leq n$.

Corollary

Like the sequence σ_*, the sequence $\Lambda_*^{\ell oc}$ is continuous along (w)-regular strata.