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SOME INVARIANTS IN REAL EQUISINGULARITY

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I. INTRODUCTION

$\mathcal{X} \subset \mathbb{C}^{n+1}$ a complex analytic hypersurface,
$e(\mathcal{X}, x)$ its local multiplicity at $x \in \mathcal{X}$.
$e(\mathcal{X}, x)$ is the simplest local analytic invariant of $\mathcal{X}$.

Zariski's definition of equisingularity (of $\mathcal{X}$ along $\mathcal{Y}$ at $y \in \mathcal{Y}$) : *Let $\mathcal{Y} \subset \mathcal{X}^{sing}$ be smooth. $\mathcal{X}$ is equisingular along $\mathcal{Y}$ at y if there exists a generic linear projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$, transverse to $\mathcal{Y}$ at y, such that, $\mathcal{C}_{\pi|\mathcal{X}}$ being the singular locus of π restricted to $\mathcal{X}^{reg}$:*

1- *If $codim_{\mathcal{X}}(\mathcal{Y}) > 1$: $\pi(\mathcal{C}_{\pi|\mathcal{X}} \cup \mathcal{X}^{sing})$ is equisingular along $\pi(\mathcal{Y})$ at $\pi(y)$.*
2- *If $codim_{\mathcal{X}}(\mathcal{Y}) = 1$: $\mathcal{Y} = \mathcal{X}^{sing}$ and $\mathcal{C}_{\pi|\mathcal{X}}$ is equimultiple (= empty !) near y.*

Some celebrated results:

Speder, 1974: Zariski's equisingularity implies Whitney's condition.

Teissier, 1973 ($\Leftarrow$) & Briançon-Speder, 1976 ($\Rightarrow$): *Whitney's condition is the same as the constancy of the Milnor numbers of generic plane sections.*

Teissier, 1981 and Henry-Merle-Sabbah, 1984: *Withney's condition is the same as Verdier's condition and is the same as equimultiplicity of the polar varieties.*

What could be a reasonable equisingularity theory in the real (algebraic, semialgebraic, subanalytic) setting ?
At least it has to start with a good **real substitute of the local complex multiplicity.**

II. A real substitute for $e(\mathcal{X}, x)$: the local density $\Theta(X, x)$

$X \subset \mathbb{R}^n$ a compact subanalytic subset of $\mathbb{R}^n$ of dimension d.

Kurdyka-Raby, 1989: For all $x \in X$, the following limit does exist:

$$\lim_{\epsilon \to 0} \frac{Vol_d(X \cap B(x, \epsilon))}{\alpha_d \cdot \epsilon^d} := \Theta(X, x),$$

with $\alpha_d = Vol_d(B(0,1))$.

Draper, 1969 and Demailly, 1987: $e(\mathcal{X}, x) = \Theta(\mathcal{X}, x)$

III. Generalization : the local Lipschitz-Killing invariants $\Lambda_i^{\ell oc}(X, x)$

By the Cauchy-Crofton formula:

$Vol_d(X) = cte(n,d) \cdot \int_{\bar{P} \in \bar{G}_n^{n-d}} \sharp(X \cap \bar{P}) \ d\bar{P}$,

where $\bar{G}_n^{n-d}$ is the Grassmann manifold of affine d-planes.

Thus we denote :

$\Lambda_i(X) := cte(n,i) \cdot \int_{\bar{P} \in \bar{G}_n^{n-i}} \chi(X \cap \bar{P}) \ d\bar{P}$, the i^{th}**-Lipschitz-Killing curvature of** X.

Theorem: For all $x \in X$, the following limit does exist:

$$\lim_{\epsilon \to 0} \frac{\Lambda_i(X \cap B(x, \epsilon))}{\alpha_i \cdot \epsilon^i} := \Lambda_i^{\ell oc}(X, x),$$

$\Lambda_i^{\ell oc}(X, x)$ is called the i^{th}**-local Lipschitz-Killing curvature of** X.

Remark: $\Lambda_0^{\ell oc}(X, x) = 1, \cdots, \Lambda_d^{\ell oc}(X, x) = \Theta(X, x), \Lambda_{d+1}^{\ell oc}(X, x) = 0, \cdots, \Lambda_n^{\ell oc}(X, x) = 0$.

IV. The polar invariants $\sigma_j(X,x)$

We consider the following functor:

$$\begin{array}{rcccl} \text{Category of compact} & & | & & \text{Category of constructible} \\ \text{subanalytic sets} & X & \longrightarrow & \mathcal{C}(X) & \text{functions on } X \\ & f\downarrow & & \downarrow f_* & \\ & Y & \longrightarrow & \mathcal{C}(Y) & \end{array}$$

where for $Z \subset X$ and $y \in Y$, $f_*(1_Z)(y) = \chi(f^{-1}(y) \cap Z)$.

The **local** equivalent diagram is:

$$\begin{array}{rcccl} \text{Category of germs of} & & | & & \text{Category of germes of} \\ \text{compact subanalytic sets} & (X,x) & \longrightarrow & \mathcal{C}(X,x) & \text{constructible functions on } X \\ & f\downarrow & & \downarrow f_* & \\ & (Y,y) & \longrightarrow & \mathcal{C}(Y,x) & \end{array}$$

where for $Z \subset X$ and $y \in Y$, $f_*(1_{(Z,x)})(y) = \chi(f^{-1}(y) \cap Z \cap B(x,r))$, r small enough.

For $f = \pi_P : (X,x) \to (P, \pi_P(x))$ a projection on the j-dimensional vector space P, we have:

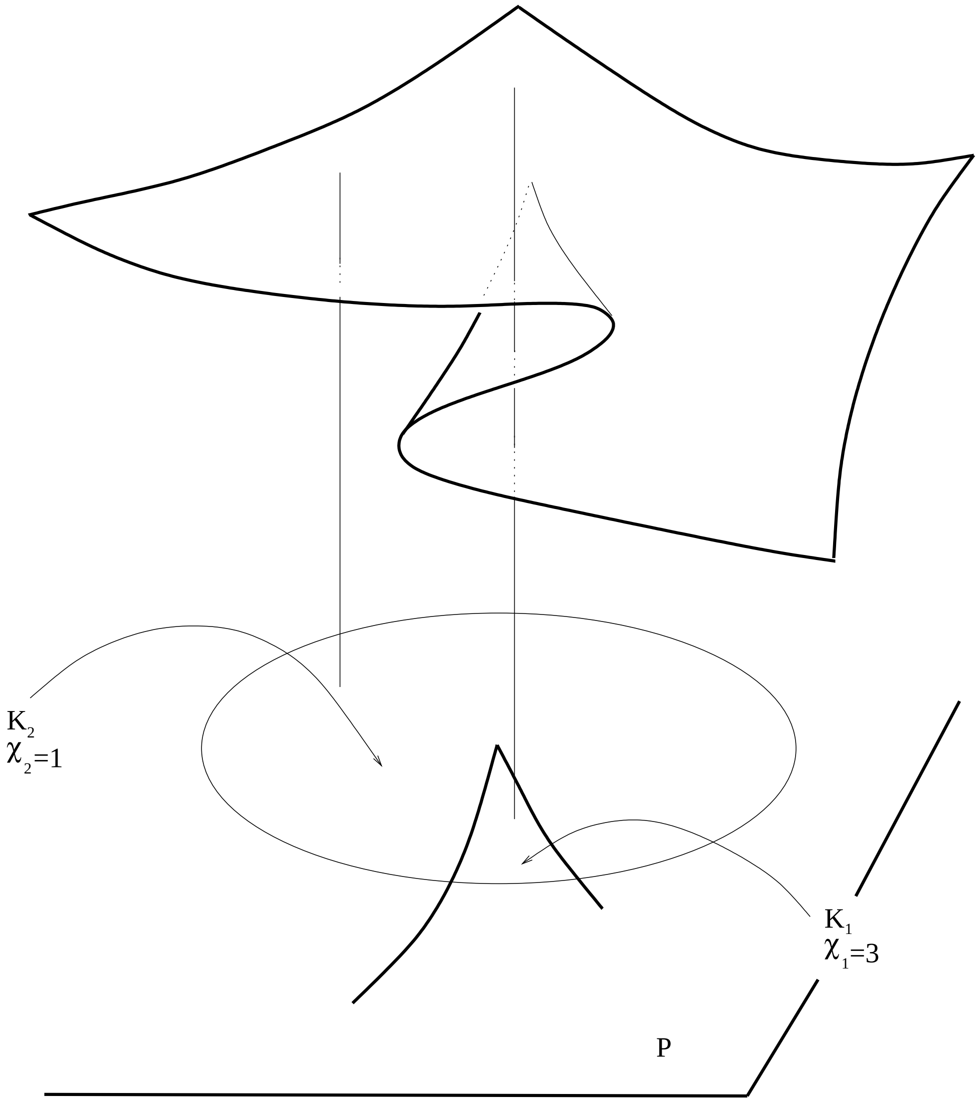
$$\begin{array}{ccc} (X,x) & \longrightarrow & \mathcal{C}(X,x) \\ \pi_P \downarrow & & \downarrow \pi_{P*} \\ (P, \pi_P(x)) & \longrightarrow & \mathcal{C}(P, \pi(p(x)) \end{array}$$

Definition (polar invariants): $\sigma_j(X,x) = \displaystyle\int_{P \in G_n^j} \int_{y \in P} \pi_{P*}(1_X)(y)\ d\Theta(y)\ dP.$

Remark: when $X = \mathcal{X}$ is an analytic hypersurface with isolated singularity in $\mathbb{C}^n$, one has:

$$\sigma_j(\mathcal{X},x) = 1 + (-1)^{n-1-j} \cdot \mu_{n-j}(\mathcal{X},x),$$

where μ_{n-j} is the Milnor number of $X \cap P$, with $P \in G_n^{n-j}$, defined by B. Teissier.



$$\sigma_j(X,x) =) \int_{P\in G_n^j} \sum_{k=1}^{k_P} \chi_k^P.\Theta(K_k^P)\ dP$$

V. Some results

Theorem 1 (real analogue of Briançon-Speder's result): Along the strata of a Verdier stratification of closed subanalytic set, the polar invariants σ_j are continuous.

Theorem 2: The following equality holds:

$$\begin{pmatrix} \Lambda_1^{\ell oc}(X,x) \\ \vdots \\ \Lambda_n^{\ell oc}(X,x) \end{pmatrix} = \begin{pmatrix} m_1^1 & m_1^2 & \dots & m_1^{n-1} & m_1^n \\ 0 & m_2^2 & \dots & m_2^{n-1} & m_2^n \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & m_n^n \end{pmatrix} \cdot \begin{pmatrix} \sigma_1(X,x) \\ \vdots \\ \sigma_n(X,x) \end{pmatrix},$$

with: $m_i^i = 1, \quad m_i^j = \dfrac{\alpha_j}{\alpha_{j-i}\cdot\alpha_i}\binom{i}{j} - \dfrac{\alpha_{j-1}}{\alpha_{j-1-i}\cdot\alpha_i}\binom{i}{j-1}, \quad \text{if} \;\; i+1 \le j \le n.$

Corollary: Along the strata of a Verdier stratification of a closed subanalytic set, the local Lipschitz-Killing invariants $\Lambda_i^{\ell oc}$ are continuous.