Zeroes and $\mathbb{Q}$-points of analytic functions

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- for $x = \dfrac{a}{b}, y = \dfrac{p}{q} \in \mathbb{Q}$ with $a \wedge b = p \wedge q = 1$,

$$\mathrm{ht}(x, y) := \max\{|a|, |b|, |p|, |q|\},$$

- $f : [0, 1] \to \mathbb{R}$ (or $f : \bar{D}(0, 1) \to \mathbb{C}$) an analytic function on a neighbourhood of $[0, 1]$ (or $\bar{D}(0, 1)$),
- Γ_f the graph of f,
- For $T \geq 1$

$$\Gamma_f(\mathbb{Q}, T) := \{(x, f(x)) \in \Gamma_f \cap \mathbb{Q}^2; \mathrm{ht}(x, f(x)) \leq T\}.$$

- for $d \in \mathbb{N}$, $\mathcal{P}_d \subset \mathbb{K}[X,Y]$ the space of polynomials of degree $\le d$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
- $$Z_d(f) := \sup_{P \in \mathcal{P}_d \setminus \{0\}} \#\{P(z, f(z)) = 0\} \in \mathbb{N} \cup \{\infty\}.$$

Remark. $Z_d(f)$ is the maximum number of intersection points between Γ_f and algebraic curves of degree $\le d$.

Call a bound for $Z_d(f)$ a *Bézout bound*.

Remark. In case f is a polynomial, $Z_d(f)$ is polynomially bounded in d (and $\deg f$) when the intersection is transverse.

Remark. By the curve selection lemma

$$Z_d(f) < \infty \text{ or } \Gamma_f \text{ contains a semialgebraic curve (of dimension 1).}$$

We assume that for any $d \in \mathbb{N}$, $Z_d(f) < \infty$, that is f is a *transcendental function*.

Theorems: $\#\Gamma_f(\mathbb{Q},T)$ is *sub-polynomial*.

- [Bombieri & Pila 1989]:

$$\forall\varepsilon>0\ \exists C_{f,\varepsilon}\ \text{ s.t. }\ \#\Gamma_f(\mathbb{Q},T)\leq C_{f,\varepsilon}T^{\varepsilon}.$$

- [Pila & Wilkie 2006]: Same result for $X\subset\mathbb{R}^n$ an o-minimal set:

$$\forall\varepsilon>0\ \exists C_{X,\varepsilon}\ \text{ s.t. }\ \#X^{\text{trans}}(\mathbb{Q},T)\leq C_{X,\varepsilon}T^{\varepsilon}.$$

$X^{\text{trans}} := X\setminus X^{\text{alg}}$,
$X^{\text{alg}} := \{x\in X; \exists S$ semialgebraic of pure dimension 1, s.t. $x\in S\subset X\}$.

Remark. [Pila 2006, Prop. 2.4]: $\Gamma_f(\mathbb{Q},T)$ is contained in a certain number of hypersurfaces of $\mathbb{R}^2$ of degree d, this number being bounded by

$$C_{f,d}T^{\varepsilon_d},$$

- $C_{f,d}$ is polynomial in d : comes from $|f^{(p)}|$.
- for $d = \lfloor \log T \rfloor$, T^{ε_d} is a constant independent of T.

Consequence. *For $f : [0,1] \to \mathbb{R}$ analytic, transcendental, $\exists Q \in \mathbb{R}[X]$ s.t.*

$$\#\Gamma_f(\mathbb{Q},T) \leq Q(\log T) \times Z_{\lfloor \log T \rfloor}(f).$$

In particular when f has a polynomial Bézout bound

$$\exists \alpha, \beta \ \ s.t. \ \ \#\Gamma_f(\mathbb{Q},T) \leq \alpha \log^\beta T \quad (vs \quad \forall \varepsilon > 0,\ \#\Gamma_f(\mathbb{Q},T) \leq \alpha C_\varepsilon T^\varepsilon)$$

Natural questions arising. *Find analytic functions $f : [0,1] \to \mathbb{R}$ with $Z_d(f)$ polynomially bounded in d.*

The overarching question

$f : \bar{D}(0, 1) \to \mathbb{C}$ transcendental analytic function

How to prescribe polynomial bounds for $Z_d(f)$?

Bézout bound of analytic functions

Remark. $Z_d(f) \leq K_d < \infty$ holds for f in any o-minimal structure
On the other hand $Z_d(f)$ may be polynomially bounded in d while f is not o-minimal (see [Gwoździewicz-Kurdyka-Parusiński 1999]).

Remark. Even when f is analytic, the asymptotic of $Z_d(f)$ is difficult to predict: for any $\zeta \in]0,1[$, there exists $f : D \to \mathbb{C}$ analytic such that for a sequence of degrees d going to ∞,

$$Z_d(f) \geq e^{d^\zeta}.$$

(see [Surroca 2002, 2006],[Pila 2004])

Remark. For f entire of finite order $:= \limsup_{r\to\infty} \dfrac{\log\log \max_{D_r} |f|}{\log r} < \infty$, for a certain sequence of degrees going to ∞

$$Z_d(f) \leq Cd^2 \qquad \text{(best possible asymptotic).}$$

What's known on the asymptotic of $Z_d(f)$?

- $f(z) = e^z$ has a polynomial Bézout bound:
  [Tijdeman 1971],
- Elementary functions have polynomial Bézout bounds:
  [Khovanskiĭ 1991],
- Entire functions with $0 <$ lower order $\leq$ finite order $< \infty$, have polynomial Bézout bounds:
  [Coman & Poletsky 2003, 2007],[Brudnyi 2008].
- Specific functions like the Riemann ζ function, the Euler Γ function, have polynomial Bézout bounds:
  [Coman & Poletsky 2007],[Masser 2011],[Besson 2011, 2014],[Boxall & Jones 2013] .
- (Compact) solutions of some algebraic differential equations have polynomial Bézout bounds:
  [Binyamini 2016].

Linear families of analytic functions

Notation.

- $\Psi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ analytic,
- $Q_1, \cdots, Q_m : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ analytic maps.

  For $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^n$, let
- $Q_\lambda(z) = \sum_{i=1}^m \lambda_i Q_i(\Psi(z)) = \sum_{k=0}^\infty v_k(\lambda) z^k$, $v_k(\lambda)$ linear forms on $\mathbb{C}^m$.

- $L_i := \{v_0 = v_1 = \cdots = v_i = 0\}$, $\mathbb{C}^m \supseteq L_0 \supseteq L_1 \supseteq \cdots \supseteq L_i \supseteq \cdots$

This sequence stabilizes at *the Bautin index $b = b_{\Psi, Q_1, \cdots, Q_m}$*:

$$L_{b-1} \supsetneq L_b = L_{b+1} = \cdots$$

Remark. $\lambda \in L_b \iff \forall k \geq 0,\ v_k(\lambda) = 0.$

Application.

- $n = 2$, $\Psi(z) = (z, f(z))$ $\;Q_i = X^j Y^p$, $\;j, p \in [\![0, d]\!]$, $\;m = (d+1)^2$,

$$\text{then } Q_\lambda(z) = \sum_{j=0}^{d} p_j(z) f^j(z), \; \deg p_j \leq d.$$

Remarks.

- The maximum number (w.r.t. λ) of zeroes of Q_λ bounds $Z_d(f)$.
- Since $\lambda \in L_b \iff \forall k \geq 0, \; v_k(\lambda) = 0$, when f is transcendental $\lambda \neq 0$ cannot cancel all v_k, therefore $L_b = \{0\}$.
- $\mathbb{C}^m \supseteq L_0 \supseteq \cdots \supseteq L_b = \{0\}$,
  therefore b minimal for $b = m - 1 = d^2 + 2d$.
- But b may be very large!

Remarks.

- For $d \geq 1$
$$b = \max_{P \in \mathcal{P}_d \setminus \{0\}} \#\{\mathrm{mult}_0 P(z, f(z))\}$$
- $(b = b_d)_{d \geq 1}$ measures the transcendency of f: the faster $(b = b_d)_{d \geq 1}$ goes to ∞ the less f seems transcendental.
- $\lim_{r \to 0} \max_{P \in \mathcal{P}_d \setminus \{0\}} \#\{z \in D_r;\ P(z, f(z)) = 0\} \leq b.$

Zeroes through Taylor coeff. and (b_d)

Notation.

- $$a_i^j := \frac{1}{i!}(f^j)^{(i)}(0)$$
- After reduction, the matrix with lines the v_k's is:

$$M = \begin{pmatrix} a_{d+1}^1 & \text{---} & a_1^1 & \cdots & a_{d+1}^d & \text{---} & a_1^d \\ | & & | & & | & & | \\ a_b^1 & \text{---} & a_{b-d}^1 & \cdots & a_b^d & \text{---} & a_{b-d}^d \end{pmatrix} \quad (1)$$

- Δ the absolute value of a nonzero $(d^2+d) \times (d^2+d)$ minor of M. (exists since $L_b = \{0\}$!)

Theorem. [C. & Yomdin 2016] *On* $D_{\frac{1}{4}}$*:*

$$Z_d(f) \le 5b \log(4 + 2(b+1)\frac{e^{2(d+1)^2 \log(d+1)}}{\Delta}).$$

Consequence. *When there exist* $R, S \in \mathbb{R}_+[X]$ *s.t.*

$$\forall d \in \mathbb{N}, \quad b \le R(d) \quad \text{and} \quad \Delta \ge e^{-S(d)},$$

$Z_d(f)$ *is polynomially bounded on* $D_{\frac{1}{4}}$.

Zeroes through Taylor coeff. and (η_d)

Notation. When $\forall k \geq 0,\ a_k = \frac{p_k}{q_k} \in \mathbb{Q}$ denote $h_\ell := \max\{|q_0|, \cdots, |q_\ell|\}$.

Proposition 1. [C. & Yomdin 2016] *For $f \in \mathbb{Q}\{z\}$, if there exist $R, S \in \mathbb{R}[X]$ s.t.*

$$b_d \leq R(d) \ \text{ and } h_\ell \leq e^{S(\ell)}$$

then $Z_d(f)$ is polynomially bounded.

Definition. f is *hypertranscendental* when f satisfies no algebraic differential equation.

Notation. For f hypertranscendental,

$$\eta_d := \max_{P \in \mathbb{Z}_d[X_0, \cdots, X_d] \setminus \{0\}} \{\mathrm{mult}_0 P(z, f(z), f'(z), \cdots, f^{(d)}(z))\}$$

Proposition 2. *For $f \in \mathbb{Q}\{z\}$, if there exist $R, S \in \mathbb{R}[X]$ s.t.*

$$\eta_d \leq R(d) \ \text{ and } h_\ell \leq e^{S(\ell)}$$

then f has a polynomial Bézout bound.

Theorem 1. [C. & Yomdin 2016] *Assume* $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{C}\{z\} \setminus \mathbb{C}[z]$, *with* $n_k^2 \leq n_{k+1} \leq n_k^q$, *for some* $q > 2$, *then*

- f is hypertranscendental ([Ostrowski 1920]).
- $b_d \leq d^{q^2}$.

  If furthermore $|a_k| \geq e^{-n_k^p}$, *for some* $p > 0$, *then*
- $Z_d(f) \leq 10(2d)^{q^2}(1 + qd^2 + 5d^{pq+3})$.

Theorem 2. (cf also [Binyamini 2016]) *Assume* $f(z)$ *is a solution of a linear differential equation with coefficients in* $\mathbb{Q}[z]$ *and rational initial conditions. Then* f *has a polynomial Bézout bound.*

Proof. b_d is polynomially bounded ([Nesterenko 88] or [Gabrielov 99]). The Taylor coefficients of f are rational and $h_\ell \leq e^{S(\ell)}$.

Random series

Notation.

- $[0,1] = I \leftarrow \cdots \leftarrow I^n \leftarrow I^{n+1} \leftarrow \cdots \leftarrow I^\infty = \varprojlim_{n\in\mathbb{N}} I^n$
- $f = \sum_{k=0}^{\infty} a_k z^k \sim (a_k)_{k\in\mathbb{N}} \in I^\infty$
- μ the measure on I^∞ induced by cylinders $\pi_n^{-1}(G), G \in I^n$ μ_n-measurable, where $\mu(\pi_n^{-1}(G)) := \mu_n(G)$.

Theorem 3. *For μ-a.e. $f \in I^\infty$, $\exists U \in \mathbb{R}[X]$, $Z_d(f) \leq U(d)$.*

Random series: Proof

Let Δ_d be the determinant of the upper $d^2 + d$ matrix of M

$$\Delta_d = \begin{vmatrix} a^1_{d+1} & \text{——} & a^1_1 & \cdots & a^d_{d+1} & \text{——} & a^d_1 \\ \Big| & & \Big| & & \Big| & & \Big| \\ a^1_{d^2+2d} & \text{——} & a^1_{d^2+d} & \cdots & a^d_{d^2+2d} & \text{——} & a^d_{d^2+d} \end{vmatrix} \qquad (2)$$

If $\Delta_d \neq 0$, b is minimal and polynomial $= d^2 + 2d$.
Δ_d is a polynomial in $f_{|I^{d^2+d}} = (a_1, \cdots, a_{d^2+2d})$, with arity $m_d := d^2 + d$ and degree $n_d : \dfrac{d(d+1)^2}{2}$.

Proposition. $\forall p \in]0,1[, \ \ \exists E_p \subset I^\infty, \ \ \mu(E_p) = p \ \ s.t.$

$$f \in E_p \Longrightarrow \forall d, \ |\Delta_d(f)| \geq e^{-(\gamma_p+3)d^5} \qquad (with\ \gamma_p \underset{p\to 1}{\sim} \log \frac{1}{1-p} \underset{p\to 1}{\to} +\infty).$$

Proof of the Theorem.

- Set $E = \bigcup_{q\in\mathbb{N}^*} E_{1-\frac{1}{q}}$. Then $\mu(E) = 1$.
- For $f \in E$, $f \in E_p$ for some p. Therefore $\Delta_d(f) \neq 0$ and $b_d = d^2 + 2d$.
- Since $|\Delta_d(f)| \geq e^{S_p(d)}$, there exists $T \in \mathbb{R}[X]$, s.t. $Z_d(f) \leq T_p(d)$.

Proof of the Proposition

Proposition. $\forall p \in]0,1[, \ \ \exists E_p \subset I^\infty, \ \ \mu(E_p) = p \ \ s.t.$

$$f \in E_p \Longrightarrow \forall d, \ \ |\Delta_d(f)| \geq e^{-(\gamma_p+3)d^5}.$$

Proof of Theorem. Fix $p \in]0,1[$.

- For $d \geq \mathbb{N}^*$, denote $\theta_d := \dfrac{1-p}{(\zeta(2)-1)d^2}$. Then $\sum_d \theta_d = 1-p$.
- $V_d := \{f \in I^\infty; \ |\Delta_d(f)| \leq \varepsilon_d\}$, where ϵ_d is maximal for $\mu(V_d) = \theta_d$.
- $E_p := I^\infty \setminus \cup_d V_d, \ \ \mu(E_p) \geq 1-(1-p) = p$, and

  $$f \in E_p \Longrightarrow \forall d, \ |\Delta_d(f)| \geq \varepsilon_d.$$
- Now let us estimate ε_d.
- By Remez inequality and since $1 \leq |\Delta_d(f)|$ for specific f:

$$1 \leq |\Delta_d|_{I^{m_d}} \leq \left(\frac{4m_d}{\mu_{m_d}(V_d)}\right)^{n_d} |\Delta_d|_{V_d} \leq \left(\frac{4m_d}{\theta_d}\right)^{n_d} \varepsilon_d$$

- It finally gives for $f \in E_p$:

$$\varepsilon_d \geq \left(\frac{\theta_d}{4m_d}\right)^{n_d} \geq \cdots \geq e^{-(\gamma_p+3)d^5}.$$

Application to $\mathbb{Q}$-points of analytic functions

$\mathbb{Q}$-Theorem. *For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, assume one of the following conditions*

1. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $b_d \leq R(d)$, $h_l \leq e^{S(l)}$,
2. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $\eta_d \leq R(d)$, $h_l \leq e^{S(l)}$,
3. $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{R}\{z\}$, $n_k^2 < n_{k+1} \leq n_k^q$, for $q > 2$, $|a_k| \geq e^{-n_k^p}$, for $p > 0$,
4. f is a solution of a linear differential equation with coefficients in $\mathbb{Q}[z]$ with rational initial conditions,
5. f is a random series,

then there exists $\alpha, \beta > 0$, s.t. $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^{\beta} T$.

Remarks.

- Statement 4 has been obtained in [Binyamini 2016].
- Statement 5 is a consequence of [Boxall & Jones 2013]: a_0 transcendent with 'good' transcendency measure, such as S-numbers in Mahler's classification (a.e. numbers are S-numbers) is enough for $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^{\beta} T$.

Application to $\mathbb{Q}$-points of analytic functions

$\mathbb{Q}$-Theorem. *For* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *assume one of the following conditions*

1. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ *s.t.* $b_d \leq R(d)$, $h_l \leq e^{S(l)}$,
2. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ *s.t.* $\eta_d \leq R(d)$, $h_l \leq e^{S(l)}$,
3. $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{R}\{z\}$, $n_k^2 < n_{k+1} \leq n_k^q$, *for* $q > 2$, $|a_k| \geq e^{-n_k^p}$, *for* $p > 0$,
4. f *is a solution of a linear differential equation with coefficients in* $\mathbb{Q}[z]$ *with rational initial conditions,*
5. f *is a random series,*

then there exists $\alpha, \beta > 0$, *s.t.* $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T$.

Remark (Siegel-Shidlovskii's theorem). A combination of comparable - but finer - conditions on f than $1, 2, 4$, generalizing Lindemann-Weierstrass theorem (f E-function), gives

$$z \neq 0, z \in \bar{\mathbb{Q}} \Longrightarrow f(z) \text{ transcendental.}$$

In particular $\#\Gamma_f(\mathbb{Q}, T) \leq 1$.