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# SOME INVARIANTS IN REAL EQUISINGULARITY

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## I. INTRODUCTION

$\mathcal{X} \subset \mathbb{C}^{n+1}$ a complex analytic hypersurface,
$e(\mathcal{X}, x)$ its local multiplicity at $x \in \mathcal{X}$.
$e(\mathcal{X}, x)$ is the simplest local analytic invariant of $\mathcal{X}$.

**Zariski's definition of equisingularity (of $\mathcal{X}$ along $\mathcal{Y}$ at $y \in \mathcal{Y}$):** *Let $\mathcal{Y} \subset \mathcal{X}^{sing}$ be smooth. $\mathcal{X}$ is* **equisingular along $\mathcal{Y}$ at** *$y$ if there exists a generic linear projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$, transverse to $\mathcal{Y}$ at $y$, such that, $\mathcal{C}_{\pi|\mathcal{X}}$ being the singular locus of $\pi$ restricted to $\mathcal{X}^{reg}$ :*

**1-** *If $codim_{\mathcal{X}}(\mathcal{Y}) > 1$: $\pi(\mathcal{C}_{\pi|\mathcal{X}} \cup \mathcal{X}^{sing})$ is equisingular along $\pi(\mathcal{Y})$ at $\pi(y)$.*

**2-** *If $codim_{\mathcal{X}}(\mathcal{Y}) = 1$: $\mathcal{Y} = \mathcal{X}^{sing}$ and $\mathcal{C}_{\pi|\mathcal{X}}$ is equimultiple (ie empty !) near $y$.*

**Whitney's $(b)$ and Kuo-Verdier's $(w)$ regularity conditions:** *Let $(Y, X)$, $Y \subset \overline{X}$ two strata of a stratification of a given set in $\mathbb{R}^n$. We say that at $y \in Y$, $(Y, X)$ is:*

$(b)$**-regular** $\Leftrightarrow$ *for any sequences $(y_i) \in Y, (x_i) \in X$ converging to $y$, assuming that the sequences $(T_{x_i}X)$ and $(x_i - y_i/\|x_i - y_i\|)$ converge to $\tau$ and $\ell$, one has: $\ell \subset \tau$.*

$(w)$**-regular** $\Leftrightarrow$ there existe $C_y$ such that locally around $y$, one has: $d(T_xX, T_yY) \leq C_y\|x - y\|$.

**Relations between $(b)$ and $(w)$:**

$(w) \underset{\not\Leftarrow}{\Rightarrow} (b)$ *for subanalytic strata (Kuo (1971), Verdier (1976)).*

$(b) \Leftrightarrow (w)$ *for complex analytic strata (Teissier, Henry-Merle (1981)).*

Some celebrated results relating **Zariski's equisingularity**, the **Whitney** or **Verdier** regularity conditions for a given stratification, and the constancy of some **numerical invariants**:

**Hironaka, 1970:** *$(b) \Rightarrow$ the constancy of $Y \ni y \mapsto e(X, y)$, for $Y$ a stratum.*

**Speder, 1974:** *Zariski's equisingularity $\Rightarrow (b)$.*

**Teissier, 1973 ($\Leftarrow$) & Briançon-Speder, 1976 ($\Rightarrow$):** *Whitney's condition is the same as the constancy of the Milnor numbers $\mu_j(\mathcal{X}, x)$ of generic plane sections.*

**Teissier, 1981 and Henry-Merle-Sabbah, 1984:** *Withney's condition is the same as Verdier's condition and is the same as equimultiplicity of the polar varieties.*

What could be a reasonable equisingularity theory in the real (algebraic, semialgebraic, subanalytic) setting ?

At least it has to start with a good **real substitute of the local complex multiplicity.**

## II. A real substitute for $e(\mathcal{X}, x)$: the local density $\Theta(X, x)$

$X \subset \mathbb{R}^n$ a compact subanalytic subset of $\mathbb{R}^n$ of dimension $d$.

**Kurdyka-Raby, 1989:** For all $x \in X$, the following limit does exist:

$$\lim_{\epsilon \to 0} \frac{Vol_d(X \cap B(x, \epsilon))}{\alpha_d \cdot \epsilon^d} := \Theta(X, x),$$

with $\alpha_d = Vol_d(B(0, 1))$. We call it the **density of $X$ at $x$**.

**Draper, 1969 and Demailly, 1987:** $e(\mathcal{X}, x) = \Theta(\mathcal{X}, x)$

## III. Generalization of the density: the local Lipschitz-Killing curvatures $\Lambda_i^{\ell oc}(X,x)$

The Cauchy-Crofton formula gives:

$$Vol_d(X) = cte(n,d) \cdot \int_{\bar{P} \in \bar{G}_n^{n-d}} \sharp(X \cap \bar{P}) \ \ d\bar{P},$$

where $\bar{G}_n^{n-d}$ is the Grassmann manifold of affine $d$-planes.

We denote :

$\Lambda_i(X) := cte(n,i) \cdot \int_{\bar{P} \in \bar{G}_n^{n-i}} \chi(X \cap \bar{P}) \ \ d\bar{P}$, the $i^{th}$-**Lipschitz-Killing curvature of** $X$.

**Another possible definition of** $\Lambda_i$**:**

$$\int_{x \in \mathbb{R}^n} \chi(X \cap B(x,r)) \ dx = \sum_{i=0}^{n} \ \alpha_i \cdot \Lambda_i(X) \cdot r^{n-i}$$

In case $X$ is convex, this is the Steiner formula:

$$Vol_n(X + B(0,r)) = \textstyle\sum_{i=0}^{n} \ \alpha_i \cdot \Lambda_i(X)$$

**Theorem:** For all $x \in X$, the following limit does exist:

$$\lim_{\epsilon \to 0} \frac{\Lambda_i(X \cap B(x,\epsilon))}{\alpha_i \cdot \epsilon^i} := \Lambda_i^{\ell oc}(X,x),$$

$\Lambda_i^{\ell oc}(X,x)$ is called the $i^{th}$-**local Lipschitz-Killing curvature of** $X$.

**Remark:**

$$\begin{array}{rcl} \Lambda_0^{\ell oc}(X,x) & = & 1 \\ & \vdots & \\ \Lambda_d^{\ell oc}(X,x) & = & \Theta(X,x), \qquad d = \dim(X) \\ \Lambda_{d+1}^{\ell oc}(X,x) & = & 0 \\ & \vdots & \\ \Lambda_n^{\ell oc}(X,x) & = & 0 \end{array}$$

## IV. The polar invariants $\sigma_j(X,x)$

We consider the following functor:

$$
\begin{array}{lcccl}
\text{Category of compact} & & | & & \text{Category of groups} \\
\text{subanalytic sets} & X & \longrightarrow & \mathcal{C}(X) & \\
 & f \downarrow & & \downarrow f_* & \\
 & Y & \longrightarrow & \mathcal{C}(Y) &
\end{array}
$$

where for $Z \subset X$ and $y \in Y$, $f_*(1_Z)(y) = \chi(f^{-1}(y) \cap Z)$.

The **local** equivalent diagram is:

$$
\begin{array}{lcccl}
\text{Category of germs of} & & | & & \text{Category} \\
\text{compact subanalytic sets} & (X,x) & \longrightarrow & \mathcal{C}(X,x) & \text{of groups} \\
 & f \downarrow & & \downarrow f_* & \\
 & (Y,f(x)) & \longrightarrow & \mathcal{C}(Y,f(x)) &
\end{array}
$$

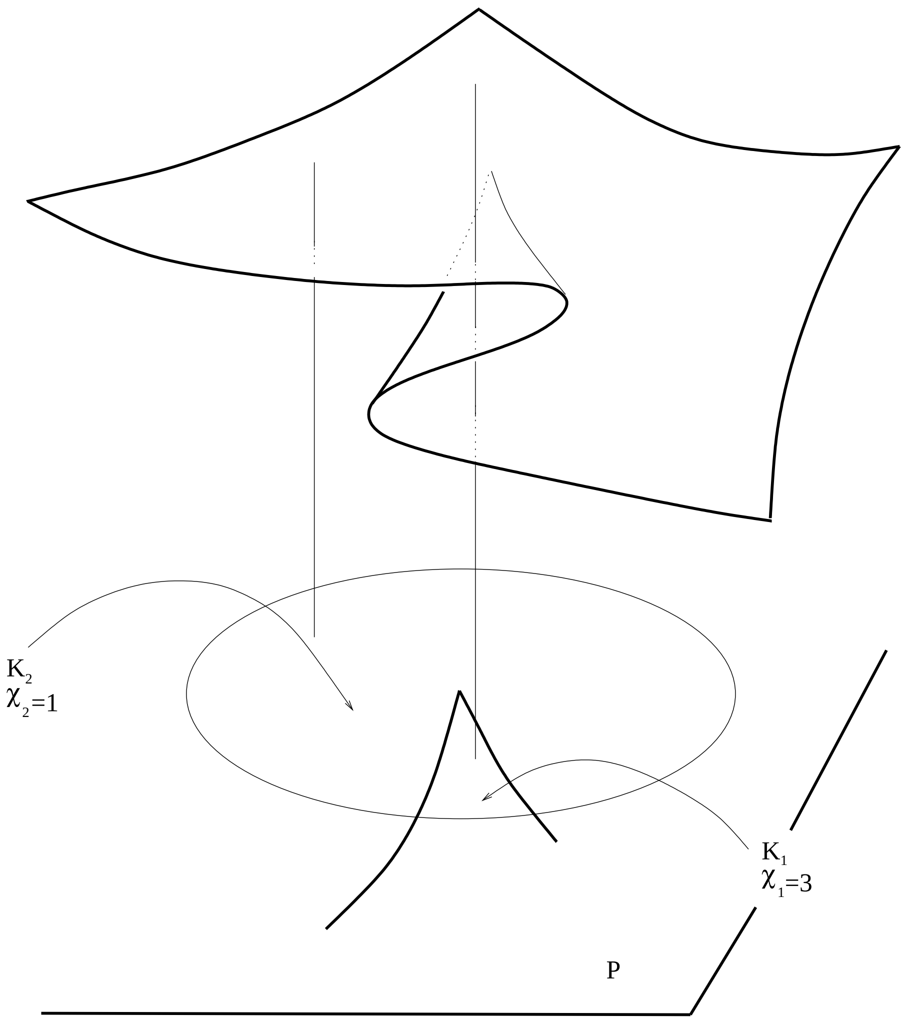
where for $Z \subset X$ and $y \in Y$, $f_*(1_{(Z,x)})(y) = \chi(f^{-1}(y) \cap Z \cap B(x,r))$, $r$ small enough.

For $f = \pi_P : (X,x) \to (P, \pi_P(x))$ a projection on the $j$-dimensional vector space $P$, we have:

$$
\begin{array}{ccc}
(X,x) & \longrightarrow & \mathcal{C}(X,x) \\
\pi_P \downarrow & & \downarrow \pi_{P*} \\
(P,\pi_P(x)) & \longrightarrow & \mathcal{C}(P,\pi(p(x)))
\end{array}
$$

**Polar invariants:** $\sigma_j(X,x) = \displaystyle\int_{P \in G_n^j} \int_{y \in P} \pi_{P*}(1_X)(y)\ d\Theta(y)\ dP.$

**Geometric interpretation**




$$\sigma_j(X,x) =) \int_{P\in G_n^j} \sum_{k=1}^{k_P} \chi_k^P.\Theta(K_k^P)\ dP$$

**Remark:** when $X = \mathcal{X}$ is an analytic hypersurface with isolated singularity in $\mathbb{C}^n$, one has:

$$\sigma_j(\mathcal{X},x) = 1 + (-1)^{n-1-j} \cdot \mu_{n-j}(\mathcal{X},x),$$

where $\mu_{n-j}$ is the Milnor number of $X \cap P$, with $P \in G_n^{n-j}$, defined by B. Teissier. The $\mu_j$ appear as coefficients in the leading term of the polynomial function: $r \mapsto length(j(f)^r/\mathbf{m} \cdot j(f))$.

## V. results

**Theorem 1 (real analogue of Briançon-Speder's result):** Along the strata of a $(w)$ stratification of closed subanalytic set, the polar invariants $\sigma_j$ are continuous.

**Theorem 2:** The $\Lambda_i^{\ell oc}$ and the $\sigma_j$ are linearly dependant:

$$\begin{pmatrix} \Lambda_1^{\ell oc}(X,x) \\ \vdots \\ \Lambda_n^{\ell oc}(X,x) \end{pmatrix} = \begin{pmatrix} 1 & m_1^2 & \dots & m_1^{n-1} & m_1^n \\ 0 & 1 & \dots & m_2^{n-1} & m_2^n \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1(X,x) \\ \vdots \\ \sigma_n(X,x) \end{pmatrix},$$

with: $m_i^j = \dfrac{\alpha_j}{\alpha_{j-i} \cdot \alpha_i} \binom{i}{j} - \dfrac{\alpha_{j-1}}{\alpha_{j-1-i} \cdot \alpha_i} \binom{i}{j-1}, \quad \text{if } \; i+1 \leq j \leq n.$

**Corollary:** Along the strata of a Verdier stratification of a closed subanalytic set, the local Lipschitz-Killing invariants $\Lambda_i^{\ell oc}$ are continuous.

**Unsolved question in convex spherical geometry** (P.McMullen-Schneider, 1983): Let $v$ be an additive $\mathbb{R}$-valued function on the compact convex sets of the $n$-sphere, which is :

- 0 on the convex sets of dimension $< n$,
- continuous with respect to the Hausdorff metric,
- invariant under rotations.

Is it true that $v = Cte \cdot Vol_n$ ?

$$\Updownarrow$$

Is any additive $\mathbb{R}$-valued function on the convex sets $\mathcal{K}$ of the sphere, continuous with respect to the Hausdorff metric and invariant under rotations (as the $\mathcal{K} \mapsto \sigma_j(\mathbb{R}_+ \cdot \mathcal{K})$ !), a linear combination of the $\mathcal{K} \mapsto \Lambda_i^{\ell oc}(\mathbb{R}_+ \cdot \mathcal{K})$ ?