

# POINTS OF BOUNDED HEIGHT ON OSCILLATORY SETS

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## Abstract

We show that transcendental curves in  $\mathbb{R}^n$  (not necessarily compact) have few rational points of bounded height provided that the curves are well behaved with respect to algebraic sets in a certain sense, and can be parametrized by functions belonging to a specified algebra of infinitely differentiable functions. Examples of such curves include logarithmic spirals and solutions to Euler equations  $x^2y'' + xy' + cy = 0$  with  $c > 0$ .

## 1. Introduction

The **height** of  $(a_1/b_1, \dots, a_n/b_n) \in \mathbb{Q}^n$  is by definition  $\max\{|a_i|, |b_i| : i = 1, \dots, n\}$ , when each pair  $(a_i, b_i) \in \mathbb{Z}^2$  is coprime. For  $X \subseteq \mathbb{R}^n$  and  $T \geq 0$ , let  $\#X(\mathbb{Q}, T)$  be the number of points in  $X \cap \mathbb{Q}^n$  of height at most  $T$ .

The overarching question: What is the asymptotic behaviour of  $\#X(\mathbb{Q}, T)$  as  $T \rightarrow +\infty$ ? As it can be notoriously difficult to determine whether  $X$  has any rational points at all, emphasis tends to be on the establishment of upper bounds. We say that  $\#X(\mathbb{Q}, T)$  is sub-polynomial if

$$\lim_{T \rightarrow +\infty} \frac{\log(\max\{1, \#X(\mathbb{Q}, T)\})}{\log T} = 0.$$

For temporary convenience in exposition, we put

$$\rho(X) = \overline{\lim}_{T \rightarrow +\infty} \frac{\log(\max\{1, \#X(\mathbb{Q}, T)\})}{\log \log T} \in [0, +\infty],$$

and say that  $X$  has finite order if  $\rho(X) < \infty$ .

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There is a natural split between the algebraic and the transcendental; we are concerned with the latter. Indeed, if  $X$  is algebraic, then  $\#X(\mathbb{Q}, T)$  is usually not sub-polynomial (see for instance Browning *et al.* [9]). On the other hand, if  $X$  is the graph of a transcendental analytic function on a compact interval of  $\mathbb{R}$ , then  $\#X(\mathbb{Q}, T)$  is sub-polynomial by Bombieri and Pila [6], and this is sharp (see Pila [25] or Surroca [30, 31]). More generally, by Pila and Wilke [28], the sub-polynomial bound holds for the ‘transcendental part’ of  $X$  (see Section 4 for the definition) if  $X$  generates an o-minimal structure on the real field (see, for example, van den Dries and Miller [14] for the definition). Several results and conjectures even assert that  $\rho(X) < +\infty$  if  $X$  has some specific dimension or is definable in some specific o-minimal structures (see Binyamini and Novikov [5], Cluckers *et al.* [11], Jones *et al.* [19], Jones and Thomas [20], Pila [24, 26, 27]). In this direction, and for more connections to logic, see Section 4.

In this paper, we do not restrict attention to non-oscillatory sets (that is, sets that generate an o-minimal structure). Rather, we develop some sufficiency conditions for showing that certain kinds of curves in  $\mathbb{R}^n$ , oscillatory or not, have finite order (see also Besson [3, 4], Boxall and Jones [7, 8], Masser [22]). Our main result is technical; a full statement is best postponed (see Theorem 2.20), but we shall give some examples now as motivation.

For  $a > 0$ , we say that a  $C^\infty$  function  $g: [a, \infty[ \rightarrow \mathbb{R}$  is **slow** (see Definition 2.2) if there exist non-negative real numbers  $A, B, C$  and  $D$  such that, for all  $x \geq a$  and  $p \in \mathbb{N}$ ,

$$\left| \frac{g^{(p)}(x)}{p!} \right| \leq D \left( A p^B \frac{\log^C x}{x} \right)^p.$$

We denote by  $S([a, +\infty[)$  the set of slow functions (with possibly different data  $A, B, C, D$ ). As observed in Remark 2.6, the set of slow functions is an algebra stable under derivation.

Let  $\mathcal{E}$  be the collection of all elementary real functions of one variable, as defined in Khovanskiĭ [21, Section 1.5], whose domain contains an unbounded-above open interval. All functions in  $\mathcal{E}$  are (real-)analytic.

**PROPOSITION 1.1** *Let  $f, g, s_1, s_2 \in (\mathcal{E} \cap S([a, +\infty[)) \cup \{\text{Id}\}$  and  $f, g$  belong to the set of functions  $h$ , satisfying*

$$h = \text{Id} \text{ or } \exists \alpha \geq 0, \quad \forall x \geq a, \quad \forall p \geq 0, \quad |h^{(p)}(x)| \leq \alpha^p.$$

*Let  $F, G > 0$ ,  $\ell, q \in \mathbb{N}^*$ , and let  $u, v$  be real numbers that are both rational numbers, or such that one of them is irrational and is not a  $U$ -number of degree  $\nu = 1$  in Mahler’s classification (see Baker [1, Chapter 8] for definitions and details, and note that almost all numbers are irrational numbers that are not  $U$ -numbers of degree  $\nu = 1$  by [1, Theorem 8.2]). Let  $x_0$  be such that the curve*

$$X := \{(u + e^{-Fx}f(s_1(x^\ell)), v + e^{-Gx}g(s_2(x^q))) : x \geq x_0\}$$

*is defined and contains no infinite semialgebraic set. Then  $X$  has finite order.*

Example of such curves is easy to produce.

EXAMPLE 1.2 The curve given by the parametrization

$$\left( \log 2 + \frac{\arctan \log^2 x}{x^5(2 + \cos^3 \log x)}, \pi + \frac{\sin \log x}{\sqrt{x}(1 + \log \log x)} \right), \quad x \geq 2$$

has finite order.

An explicit bound on  $\rho$  can be computed in terms of the complexity of the elementary functions  $f$ ,  $g$ ,  $s_1$  and  $s_2$ . Thus, for particular choices of  $f$ ,  $g$ ,  $s_1$ ,  $s_2$ , we can get more refined results, such as:

PROPOSITION 1.3 *Let  $a_1, a_2, F, G, c_1, c_2$  be non-zero real numbers with  $F, G > 0$ . Let  $\ell, q \in \mathbb{N}^*$  and let  $u, v$  be as in Proposition 1.1. Then*

$$X := \{(u + a_1 e^{F x} \cos(c_1 x^\ell), v + a_2 e^{G x} \sin(c_2 x^q)) : x \in \mathbb{R}\}$$

has order at most  $5 + 4 \max\{\ell, q\}$ .

As a special case, we have  $\rho(\mathbb{S}_\omega) \leq 9$  for every logarithmic spiral

$$\mathbb{S}_\omega := \{(e^x \cos(\omega x), e^x \sin(\omega x)) : x \in \mathbb{R}\} \quad (\omega \neq 0).$$

Observe that  $\mathbb{S}_\omega$  is a maximal (real-time) trajectory of  $\dot{z} = (1 + i\omega)z$  and a subgroup of  $(\mathbb{C}^*, \cdot)$ . It is easy to see that  $\rho(\mathbb{S}_{\pi/\log 2}) \geq 1$ .

We have a related result for functions.

PROPOSITION 1.4 *Let  $f, s \in (\mathcal{E} \cap S([a, +\infty[)) \cup \{\text{Id}\}$ , and  $\ell$  and  $f$  be as in Proposition 1.1. Then the graph,  $X$ , of  $f \circ s \circ \log^\ell$  has finite order. In particular, let  $a$  and  $c$  be non-zero real numbers,  $\ell \in \mathbb{N}^*$  and  $f \in \{\sin, \cos\}$ . Then, the graph of  $af(c \log^\ell)$  has order at most  $5 + 4\ell$ .*

With  $\ell = 1$ , we obtain maximal solutions to the Euler equation  $x^2 y'' + xy' + c^2 y = 0$ , indeed,  $\{\cos(c \log), \sin(c \log)\}$  is a set of fundamental solutions on  $]0, \infty[$ . It is easy to see that  $\rho \geq 1$  for  $c = \pi/\log 2$ . The proof generalizes to show that all transcendental slow functions in  $\mathcal{E}$  that satisfy certain further technical conditions have finite order (see Proposition 3.6).

To the best of our knowledge, these results yield the first known examples of real-analytic submanifolds (embedded and connected) of the plane having non-zero finite order and whose intersection with some real-algebraic set has infinitely many connected components. (By Lindemann–Weierstrass,  $\rho(\sin x) = 0$ . It is an easy exercise that  $\rho(\sin(\pi x)) = +\infty$ .)

For  $X$  as in Propositions 1.1, 1.3 and 1.4, it follows from the work of Pila [24] that every compact subset of  $X$  has finite order, but via proof that does not yield finite order for  $X$  itself. We remedy this by establishing the existence of compact connected  $K_X \subseteq X$  such that  $\rho(X \setminus K_X) < \infty$ , thus yielding  $\rho(X) = \max(\rho(K_X), \rho(X \setminus K_X)) < +\infty$ . The constants witnessing the bounds appearing in [24] depend *a priori* on the choice of the compact subset of  $X$ , and in no case could we conclude in some visibly simple way that the set of all such constants may be bounded as the length of our parametrizing interval goes to infinity. For instance, by [24], every bounded subset of the graph of  $\sin \pi x$  has finite order, but again  $\sin \pi x$  has infinite order on any unbounded interval. Thus, dealing with the oscillatory case requires tight control of any involved constants appearing in

any known methods as the parameter interval we are looking at increases in length. Since we use here the determinant method of [6], as in [28], and since the constants that arise from this method come from bounds on the derivatives of the functions parametrizing the set, we impose some tameness on the derivatives via the algebra of slow functions. This condition may be viewed as the oscillatory and non-compact counterpart of Pila's mild parametrizations; this somewhat answers the wish of [24, 4.3 Remark 3]. One then obtains in Theorem 2.20 and Proposition 3.6 general explicit bounds for  $\#X(\mathbb{Q}, T)$  in a split form of a product of two factors, reflecting two features of different nature of  $X$ : one arising from the slowness hypothesis; the other from how often our curve intersects algebraic curves of given degrees.

There are non-oscillatory unbounded functions of finite non-zero order; as examples, the function  $2^x$  has order 1 (indeed, the only rational points are  $(k, 2^k)$ ,  $k \in \mathbb{Z}$ ) and the restriction to the positive real line of the Euler gamma function  $\Gamma$  has order at most 2 (see [8]) and at least 1 (consider the values at positive integers and recall Stirling's formula). The restriction to  $]1, \infty[$  of the Riemann zeta function  $\zeta$  has order at most 2 (again see [8]), but it is not known if the order is positive. Our methods are not confined to the oscillatory case; for illustrative purposes, we shall give alternate proofs of the finiteness of the orders of  $\Gamma$  and  $\zeta$  (see 'The Riemann  $\zeta$  function' and 'The Euler  $\Gamma$  function' sections), though our bounds are weaker than those already known.

## 2. Counting rational points on curves

For  $X \subseteq \mathbb{R}^n$  and  $T \geq 0$ , we let  $X(\mathbb{Q}, T)$  denote the set of rational points in  $X$  with height  $\leq T$ . Let  $\Gamma \subset \mathbb{R}^n$  be a parameterized curve, that is, the image of  $n$  smooth (that is, infinitely differentiable) functions  $\gamma = (f_1, \dots, f_n): I \rightarrow \mathbb{R}^n$ , for  $I$  an interval in  $\mathbb{R}$ , such that  $\gamma(I) = \Gamma$ .

The goal of this section is to provide, under some hypothesis on the derivatives of the  $f_i$ , a bound for  $\#\Gamma(\mathbb{Q}, T)$ . Evidently, it is equivalent to provide such a bound on the projection of  $\Gamma$  onto some coordinate plane of  $\mathbb{R}^n$ ; from now on, we can assume without loss of generality that our forthcoming assumptions on the  $f_i$  concern only two coordinate functions among the  $f_i$ , or more conveniently, that  $n = 2$ . Therefore, in the sequel we let  $\gamma = (f, g)$  be a parametrization of a given curve  $\Gamma \subset \mathbb{R}^2$ .

For  $N, L \in \mathbb{R}^*$  and an interval  $J \subset \mathbb{R}$ , we put

$$\Gamma_{N,L} := \gamma([N, N + L]),$$

$$\Gamma_{N,+\infty} := \gamma([N, +\infty[),$$

$$\Gamma_J := \gamma(J).$$

**LEMMA 2.1** *Fix  $\mu \in \mathbb{N}^*$ . Let  $I$  be an interval of length  $L$ . Let  $x_1, \dots, x_\mu$  be points in  $I$  and  $\psi_1, \dots, \psi_\mu$  be  $C^{\mu-1}$ -functions from  $I$  to  $\mathbb{R}$ . Set*

$$\Delta = \det(\psi_i(x_j)).$$

*Then*

$$|\Delta| \leq L^{\frac{\mu(\mu-1)}{2}} \sum_{s \in S(0, \dots, \mu-1)} \sigma_{1,s(0)} \sigma_{2,s(1)} \cdots \sigma_{\mu,s(\mu-1)},$$

where  $S(0, \dots, \mu - 1)$  is the permutation group of  $\{0, \dots, \mu - 1\}$  and

$$\sigma_{i,p} = \sup_{\xi \in I} \frac{1}{p!} |\psi_i^{(p)}(\xi)|, \quad p \in \{0, \dots, \mu - 1\}.$$

*Proof.* We use the by-now classical bound for  $|\Delta|$  (see [6, Proposition 2], [24]):

$$|\Delta| \leq |V(x_1, \dots, x_\mu)| \cdot \left| \det \left( \frac{\psi_j^{(i-1)}(\xi_{ij})}{(i-1)!} \right)_{i,j=1, \dots, \mu} \right|,$$

where  $V$  is the Vandermonde determinant and  $\xi_{ij}$  are points in  $I$ . The statement follows.  $\square$

In order to bound from above the determinant  $\Delta$ , we consider from now on a specific bound condition on the derivatives of our parametrization  $\gamma$ . This condition will allow for control of the maximal length  $L$  of a parameter interval  $[x, x + L]$  such that  $\Gamma_{x,L}(\mathbb{Q}, T)$  is contained in a single algebraic curve of given degree  $d$  (see Proposition 2.13), and thus to eventually control the number of such algebraic curves needed to cover  $\Gamma(\mathbb{Q}, T)$  (see Proposition 2.17).

**DEFINITION 2.2** Let  $a > 1$ . We say that a smooth parametrization

$$\gamma = (f, g): [a, +\infty[ \rightarrow \mathbb{R}^2$$

of a bounded curve  $\Gamma$  is a **slow parametrization** of  $\Gamma$  (with data  $A, B, C, u$  and  $b$ ) if:

(1) there exist  $u \in \mathbb{R}$  and  $b: [a, +\infty[ \rightarrow \mathbb{R}_+$  decreasing to zero such that, for all  $x \geq a$ ,

$$|f(x) - u| \leq b(x);$$

(2) there exist non-negative real numbers  $A, B, C$  such that for all  $p \geq 0$  and  $x \geq a$ ,

$$\left| \frac{f^{(p)}(x)}{p!} \right| \leq \varphi_p(x), \quad \left| \frac{g^{(p)}(x)}{p!} \right| \leq \varphi_p(x),$$

where

$$\varphi_p(x) = \left( Ap^B \frac{\log^C x}{x} \right)^p.$$

We call a function satisfying condition (2) above a **slow function**, and say that the set of constants  $A, B, C$  is **attached** to the function, or attached to the slow parametrization  $\gamma$ .

REMARK 2.3 In full generality one should rather define  $\varphi_p$  as

$$\varphi_p(x) = D \left( Ap^B \frac{\log^C x}{x} \right)^p, \quad (2.1)$$

for some real number  $D$  (as in the introduction) in order to have stability by addition for the set of slow functions, as pointed out in Remark 2.6 below. But the condition  $f \leq 1, g \leq 1$ , made for simplicity in forthcoming computations, is made without loss of generality, up to dividing  $f$  and  $g$  by an integer bounding  $f$  and  $g$ , which is harmless for counting rational points.

REMARK 2.4 In our Definition 2.2, the first coordinate  $f$  of a slow parametrization  $\gamma$  goes to  $u$  as the parameter goes to infinity, at speed at least  $b$ . In the sequel, for applications, we will often consider the case  $b(x) = 1/x^E$ , with  $E > 0$ , and the Diophantine properties of  $u$  have to be considered for counting rational points on  $\Gamma$ ; see Remark 2.15(4).

REMARK 2.5 If a function  $k$  (bounded from above or not) is greater than 1 and satisfies condition (2) of Definition 2.2 for  $p \geq 1$ , then  $1/k$  is slow.

REMARK 2.6 For  $a > 1$ , let us denote by  $S([a, +\infty[)$ , the set of slow functions defined on  $[a, +\infty[$  (with the bounding function  $\varphi_p$  as in (2.1), and with possible different data  $A, B, C, D, u, b$  for each of them). An easy computation, essentially based on the formula (2.2) given below, shows that  $S([a, +\infty[)$  is a subalgebra of  $C^\infty([a, +\infty[)$ , stable under derivation.

REMARK 2.7 For a given slow parametrization, we can always assume that  $A, B, C$  are large enough, up to quantitatively weakening the condition of being a slow parametrization. The condition that the parameter has to be larger than  $a$  in this definition is technical. On one hand, we are essentially interested in the part of the curve corresponding to a neighbourhood of infinity of the parameter, because this part of the curve is the oscillatory part when the parametrization is analytic. On the other hand, for analytic curves  $\Gamma$  restricted to a compact interval of parameters, since we have a mild parametrization (see [24]), one gets a bound for  $\#\Gamma(\mathbb{Q}, T)$  as given in Theorem 2.20, depending on how much  $\Gamma$  cuts algebraic curves of given degree, or at least we have a general sub-polynomial bound for  $\#\Gamma(\mathbb{Q}, T)$ , provided by [28] for the o-minimal context.

REMARK 2.8 The bound (2) in Definition 2.2 may be seen as the unbounded version of the mild parametrization introduced in [24]. If an analytic function  $g: \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfies the bound (2) of Definition 2.2 with  $B = 0$ , then there is a complex analytic continuation of  $g$  to the half plane  $\{x \in \mathbb{C}, \Re(z) > 0\}$ , since the radius of convergence of the Taylor series of  $g$  at  $x$  is greater than  $\frac{x}{A \log^C x}$ .

REMARK 2.9 A slow parametrization may also be thought of as the oscillatory version of Yomdin–Gromov parametrization for o-minimal sets (see Gromov [16], Yomdin [34] and [28]), since a slow parametrization will provide, for fixed  $T$  and  $d$ , a finite number  $n(T)$  of intervals  $I$  such that  $\Gamma_I(\mathbb{Q}, T)$  is contained in a single algebraic curve of degree  $d$ . In the o-minimal case, since the number  $n(T)$  given by a Yomdin–Gromov parametrization is not controlled, one cannot expect better bounds for  $\#\Gamma(\mathbb{Q}, T)$  than  $\alpha_\epsilon T^\epsilon$  (for any  $\epsilon > 0$ ). Contrariwise, here, the fast decay on the derivatives imposed by a slow parametrization will provide a sufficiently small number  $n(T)$  (Lemma 2.17) that will in turn imply that  $\Gamma$  has finite order, provided that good enough

Bézout bounds (Definition 2.18) are available (Theorem 2.20). It is worth noting that, starting with a given slow parametrization, the constants  $\alpha$  and  $\beta$  in our bound  $\alpha \log^\beta T$  for  $\#\Gamma(\mathbb{Q}, T)$  in any of Theorem 2.20, Proposition 3.4 or 3.6 will depend explicitly on the data  $A, B, C$  and  $b$ , though the density of rational points in the set  $\Gamma$  should be independent of any choice of parametrization. This obvious constraint theoretically explains why, in practice, the range of slow parametrizations, and reparametrizations that improve the decay of a given slow parametrization or the decay of  $b$ , is necessarily limited. When one composes a slow parametrization with a smooth increasing bijection  $\sigma: [a', +\infty[ \rightarrow [a, +\infty[$  having smaller and smaller derivatives as  $x$  goes to infinity (for example,  $x \mapsto \log x$ ), the decay of the derivatives of the reparametrization should improve. But at the same time, since  $\sigma$  is contracting the distance and the slow parametrization we started with has smaller and smaller derivatives as  $x$  gets larger and larger, this effect is counterbalanced. It thus appears that any given slow parametrization is a quite stable and optimal form of parametrization with respect to reparametrization and counting rational points of bounded height in a curve.

We assume in what follows that  $\gamma = (f, g)$  is a slow parametrization of the curve  $\Gamma$ . Now let us choose  $d \in \mathbb{N}^*$  and let us denote  $(\alpha_1, \alpha_2) \in \mathbb{N}^2$  by  $\alpha$  and the function  $f^{\alpha_1} g^{\alpha_2}$  by  $\gamma^\alpha$ .

In the following lemmas, we will use the classical formula

$$\frac{(f^\alpha g^\beta)^{(p)}}{p!} = \sum_{i_1 + \dots + i_\alpha + j_1 + \dots + j_\beta = p} \frac{f^{(i_1)} \dots f^{(i_\alpha)} g^{(j_1)} \dots g^{(j_\beta)}}{i_1! \dots i_\alpha! j_1! \dots j_\beta!}, \quad \alpha, \beta, p \in \mathbb{N}. \tag{2.2}$$

LEMMA 2.10 *Let  $\gamma$  be a slow parametrization of a plane curve  $\Gamma$ , with constants  $A, B, C$ . Then for any  $\alpha \in \mathbb{N}^2$  and  $p \in \mathbb{N}$ ,*

$$\frac{|(\gamma^\alpha)^{(p)}(x)|}{p!} \leq (p + 1)^{\alpha_1 + \alpha_2} A^p p^{Bp} \frac{\log^{Cp} x}{x^p}.$$

(Hence,  $\gamma^\alpha \in S([a, +\infty[.)$ )

*Proof.* By formula (2.2),

$$\begin{aligned} \frac{|(\gamma^\alpha)^{(p)}(x)|}{p!} &\leq \sum_{i_1 + \dots + i_{\alpha_1} + j_1 + \dots + j_{\alpha_2} = p} A^p i_1^{B i_1} \dots i_{\alpha_1}^{B i_{\alpha_1}} j_1^{B j_1} \dots j_{\alpha_2}^{B j_{\alpha_2}} \frac{\log^{Cp} x}{x^p} \\ &\leq (p + 1)^{\alpha_1 + \alpha_2} A^p p^{Bp} \frac{\log^{Cp} x}{x^p}. \end{aligned} \quad \square$$

We now recall another bound that will be used in the proof of Proposition 2.12.

LEMMA 2.11 *For any  $\mu \geq 1$  and  $B > 0$ ,*

$$\prod_{r=0}^{\mu-1} r^{Br} \leq \mu^{B \frac{\mu(\mu-1)}{2}}.$$

*Proof.* We use the classical bound for such a product (see also [24, Proposition 2.2]). We have

$$\begin{aligned} \log \left( \prod_{r=0}^{\mu-1} r^{Br} \right) &= B \sum_{r=1}^{\mu-1} r \log r \leq B \int_1^{\mu} t \log t \, dt \\ &\leq B \frac{\mu(\mu-1) \log \mu}{2}. \end{aligned} \quad \square$$

Let us now fix some integer  $d \geq 1$  and set

$$\mu = \frac{(d+2)(d+1)}{2} = \#\{\alpha \in \mathbb{N}^2: |\alpha| \leq d\}, \quad \rho = \frac{\mu(\mu-1)}{2},$$

and for  $\mu$  points  $x_1, \dots, x_\mu$  given in the domain of definition of  $\gamma$ , consider the determinant

$$\Delta = \det(\gamma^\alpha(x_j))_{\alpha \in \Delta_2(d), j \in \{1, \dots, \mu\}}.$$

**PROPOSITION 2.12** *Let  $\gamma$  be a slow parametrization of a plane curve  $\Gamma$ . With the above notation, for any  $d \geq 1$ , there exist  $N = N(C) \geq 1$  and  $C(d, A, B) > 0$  such that, for any  $L > 0$  and  $x_1, \dots, x_\mu \in [N; N + L]$ ,*

$$|\Delta| \leq C(d, A, B) L^\rho \frac{\log^{C\rho} N}{N^\rho}.$$

Furthermore, one can take  $C(d, A, B) = \mu! A^\rho \mu^{d(\mu-1)} \mu^{B\rho}$ .

*Proof.* Let us first note that for any  $p = 0, \dots, \mu - 1$ ,  $\log^{Cp} x / x^p$  is decreasing on  $[N, +\infty[$  with, for instance,  $N = N(C) = e^C$ . Combining Lemmas 2.1, 2.10 and these two remarks, one obtains for  $x_1, \dots, x_\mu \in [N, N + L]$

$$\begin{aligned} |\Delta| &\leq L^{\frac{\mu(\mu-1)}{2}} \sum_{s \in S(0, \dots, \mu-1)} A^{\sum_{j=0}^{\mu-1} j} \prod_{r=0}^{\mu-1} r^{Br} \prod_{r=0}^{\mu-1} (r+1)^d \frac{\log^{C \sum_{j=0}^{\mu-1} j} N}{N^{\sum_{j=0}^{\mu-1} j}} \\ &\leq A^\rho \mu^{d(\mu-1)} L^\rho \frac{\log^{C\rho} N}{N^\rho} \sum_{s \in S(0, \dots, \mu-1)} \prod_{r=0}^{\mu-1} r^{Br}. \end{aligned}$$

Now using Lemma 2.11, we have

$$|\Delta| \leq \mu! A^\rho \mu^{d(\mu-1)} \mu^{B\rho} L^\rho \frac{\log^{C\rho} N}{N^\rho}. \quad \square$$

Let us now fix  $T \geq 1$  and assume that the points  $x_1, \dots, x_\mu \in [N, N + L]$ , with the notation of Proposition 2.12, are such that  $(f(x_j), g(x_j))$ ,  $j = 1, \dots, \mu$ , is a pair of rational points of height  $\leq T$ . In this case, if  $\Delta \neq 0$  we have,

$$|\Delta| \prod_{j=1, \dots, \mu} |a_j^d b_j^d| \geq 1,$$

where  $a_j$  and  $b_j$  are the denominator of  $f(x_j)$  and  $g(x_j)$ , respectively. But since

$$\prod_{j=1, \dots, \mu} |a_j^d b_j^d| \leq T^{2d\mu},$$

we also have, again if  $\Delta \neq 0$ ,

$$T^{2d\mu} |\Delta| \geq 1.$$

Now, considering Proposition 2.12, as soon as

$$T^{2d\mu} C(d, A, B) L^\rho \frac{\log^{C\rho} N}{N^\rho} < 1, \quad (2.3)$$

we necessarily have  $\Delta = 0$ . Let us fix  $L$  as

$$L := L(d, A, B, T, N) = C'(d, A, B) \frac{N}{\log^C N} T^{-\frac{4d}{\mu-1}}, \quad (2.4)$$

with  $C'(d, A, B) = C(d, A, B)^{\frac{1}{\mu}}$ . It follows that for any interval  $[N; N + L[$ , with  $N \geq N(C)$  given by Proposition 2.12, and  $L$  given by equation (2.4), for any choice of points  $x_1, \dots, x_\mu \in [N; N + L[$  and such that  $\gamma(x_j) \in I_{N,L}(\mathbb{Q}, T)$ , the rank of  $(\gamma^\alpha(x_j))_{\alpha \in \Delta_2(d), j=1, \dots, \mu}$ , is  $\leq \mu - 1$ .

We now proceed similar to [6, Lemma 1]. Let  $a$  be the maximal rank of  $(\gamma^\alpha(x_j))$  over all  $x_1, \dots, x_\mu \in [N; N + L[$  such that  $\gamma(x_j) \in I_{N,L}(\mathbb{Q}, T)$  and let  $M = (\gamma^\alpha(x_j))_{\alpha \in I, j \in \{1, \dots, a\}}$  be of rank  $a$ , for some fixed  $x_1, \dots, x_a \in [N; N + L[$  such that  $\gamma(x_j) \in I_{N,L}(\mathbb{Q}, T)$  and some  $I \subset \{\alpha \in \mathbb{N}^2, |\alpha| \leq d\}$  of cardinality  $a$ . Since  $a < \mu$ , we can choose  $\beta = (\beta_1, \beta_2) \in \Delta_2(d) \setminus I$ .

Let us denote by  $f(y) \in \mathbb{R}[y]$ , the determinant of the matrix

$$(M' y^\delta)_{\delta \in I \cup \{\beta\}},$$

where  $y = (y_1, y_2)$ ,  $y^\delta = y_1^{\delta_1} y_2^{\delta_2}$ , and  $M'$  is  $M$  augmented by the line  $\gamma^\beta(x_j)_{j \in \{1, \dots, a\}}$ . Then  $f(y)$  is a polynomial with real coefficients that is not zero, since the coefficient of the monomial  $y^\beta$  in  $f(y)$  is the non-zero minor  $\det(M)$ , and the degree of  $f(y)$  is at most  $d$ . Furthermore, for any  $(x_1, x_2) \in I_{N,L}(\mathbb{Q}, T)$ , we have  $f(x_1, x_2) = 0$ , by definition of the maximal rank  $a$ .

Note that in case there are fewer than  $\mu$  points  $x$  in  $[N, N + L]$  such that  $\gamma(x) \in I_{N,L}(\mathbb{Q}, T)$ , those points  $\gamma(x)$  are certainly in some algebraic curve of degree less than  $d$ , since the dimension of the space of polynomials of  $\mathbb{R}[y]$  of degree at most  $d$  is  $\mu$ .

To sum up this discussion, we have proved the following statement.

**PROPOSITION 2.13** *Let  $\gamma$  be a slow parametrization of a plane curve  $\Gamma$ . Let  $d \in \mathbb{N}^*$  and  $T \geq 1$ . There exists  $N(C) \geq 1$  such that for any  $N \geq N(C)$  exists  $L = L(d, A, B, T, N)$ , such that the*

set  $\Gamma_{N,L}(\mathbb{Q}, T)$  is contained in an algebraic curve of  $\mathbb{R}^2$  of degree at most  $d$ . Furthermore, one can take  $L = C'(d, A, B) \frac{N}{\log^c N} T^{-\nu}$ , with  $\nu = \frac{4d}{\mu-1}$  and  $C'(d, A, B) = A^{-1} \mu^{-B} \mu^{\frac{-2d}{\mu}} (\mu!)^{\frac{-2}{\mu(\mu-1)}}$ .

So far, we have introduced the notion of slow parametrization and given bounds for such parametrization, like in Proposition 2.13. This is in order to eventually bound the number of rational points of bounded height in the range  $\Gamma \subset \mathbb{R}^2$  of the slow parametrization we consider. At this point, one may have the naive idea that the slower this parametrization is (that is, the smaller  $A, B$  and  $C$  are), the better this bound should be, since the slower our parametrization is, the larger the length  $L$  provided by Proposition 2.13 is, and thus the smaller the number  $n(T)$  of intervals  $I$  such that  $\Gamma_I(\mathbb{Q}, T)$  is contained in one algebraic curve of given degree is. But of course the density in  $\Gamma$  of rational points of given height does not depend on the parametrization of  $\Gamma$  that maybe could be taken slower than it is, up to a convenient reparametrization of the half line by itself. There is no contradiction here. In fact such a reparametrization also increases the time needed to parametrize the rational points of height less than  $T$ , and this growth balances the gain obtained on the bounds of the derivatives after reparametrization. Nevertheless, we believe that this balancing effect must be made visible in our final bound, with the hope that one can, in a particular situation, optimize this balance. For this goal, one introduces a control of the speed at which a slow parametrization, or a suitable change of variables made to obtain a slower parametrization, runs through the rational points of given bounded height. On the other hand, so far, we have not mentioned that in order to obtain a bound on  $\#\Gamma(\mathbb{Q}, T)$  from assumptions on a parametrization of  $\Gamma$ , one has to exclude some non-injective behaviour of the parametrization under consideration: passing too many times by the same rational point leaves no possibility to bound  $\#\Gamma(\mathbb{Q}, T)$  throughout a parametrization. The next definition proposes such a notion of quantitative control on the time needed to go through all the points of given bounded height as well providing a quantitative control of, let us say, the non-injectivity of our parametrization. The better is this control, called a height control function, the better will be our final bounds. It is thus worth considering as a separate additional assumption such a control function and state our bounds in terms of this given height control function.

DEFINITION 2.14 Let  $\gamma: [a, +\infty[ \rightarrow \mathbb{R}^2$  be a parametrization of a plane curve  $\Gamma$ . We say that a function  $\varphi: [a, +\infty[ \rightarrow \mathbb{R}$  is a **height control function** for  $\gamma$  if, for all  $T \geq 1$ ,

$$\gamma^{-1}(\Gamma(\mathbb{Q}, T)) \subset [a, \varphi(T)].$$

REMARK 2.15 In the following particular cases, one can easily compute a height control function of  $\gamma = (f, g)$ . The preliminary observation is useful: for  $a, p \in \mathbb{Q}, b, q \in \mathbb{Q}^*, a/b - p/q = 0$  or  $|a/b - p/q| \geq K/|q|$ , for  $K = 1/|b|$ .

- (1) In case that  $f$  and  $g$  are decreasing to  $u, v \in \mathbb{Q}$ , a height control function of  $\gamma$  is given by  $\varphi(T) = \min\left(f^{-1}\left(\frac{K}{T}\right), g^{-1}\left(\frac{K}{T}\right)\right)$ , for some  $K > 0$ .
- (2) If  $u, v \in \mathbb{Q}$  and  $|f - u|$  and  $|g - v|$  are, respectively, bounded from above by functions  $b$  and  $c$  that decrease to 0, a height control function of  $\gamma$  is given by  $\varphi(T) = \max\left(b^{-1}\left(\frac{K}{T}\right), c^{-1}\left(\frac{K}{T}\right)\right)$ ,

for some  $K > 0$ , as soon as  $f - u$  and  $g - v$  have no common zero on  $[a, +\infty[$  (note that contrariwise to (1), in this case the functions  $f - u$  or  $g - v$  may have zeroes between  $\min\left(b^{-1}\left(\frac{K}{T}\right), c^{-1}\left(\frac{K}{T}\right)\right)$  and  $\max\left(b^{-1}\left(\frac{K}{T}\right), c^{-1}\left(\frac{K}{T}\right)\right)$ ).

- (3) In case  $u \in \mathbb{Q}$ , any decreasing function  $b: [a, +\infty[ \rightarrow \mathbb{R}$  going to 0 and bounding  $|f - u|$  gives rise to a height control function of  $\gamma$  defined by  $\varphi(T) = b^{-1}\left(\frac{K}{T}\right)$ , for some  $K > 0$ , as soon as  $f$  does not take the value  $u$ . In particular, when  $\gamma$  is a slow parametrization as in Definition 2.2, with  $u \in \mathbb{Q}$  and  $b(x) = 1/x^E$ ,  $E > 0$ , one can take  $\varphi(T) = T^{\frac{K}{E}}$ , for some  $K > 0$ , as soon as  $f$  does not take the value  $u$  on  $[a, +\infty[$ .
- (4) In case  $u \notin \mathbb{Q}$ , let us denote by  $\tau$  an **irrationality measure function** of  $u$ , that is a function such that for any  $p, q \in \mathbb{N}$  with height less than  $T$ ,  $\left|u - \frac{p}{q}\right| \geq e^{-\tau(T)}$ . When  $(f, g)$  is slow, since  $|f(x) - u| \leq b(x)$  and  $b$  decreases to 0, a height control function for  $\gamma$  is given by  $b^{-1}(e^{-\tau(T)})$ . For instance by Roth's theorem [29], in case  $u$  is an algebraic number,  $\tau$  may be  $\log \frac{T^3}{C}$ , where  $C > 0$  depends on  $u$ . In case  $u = \pi$  (by Cijssouw's theorem [10]), or in case  $u = \log w$ , for  $w$  an algebraic number not 0 or 1,  $\tau$  may be  $K \log T$  (see [1, Chapter 3]). In fact  $\tau(T) = K \log T$ , and thus  $\varphi(T) = b^{-1}(1/T^K)$ , is possible for any number which is not a  $U$ -number of degree  $\nu = 1$  in Mahler's classification, and almost all numbers being  $S$ -numbers, almost all numbers are not  $U$ -numbers (see [1, Chapter 8]).

For fixed  $T \geq 1$ ,  $d \in \mathbb{N}^*$  and with the notation introduced in Proposition 2.13, we define a sequence  $(x_n)_{n \in \mathbb{N}}$  of real numbers by

$$x_0 = N(C), \quad x_{n+1} = x_n + C'(d, A, B) \frac{x_n}{\log^C x_n} T^{-\nu}.$$

Then,  $[x_n, x_{n+1}]$  is an interval in  $[N(C), +\infty[$  of length

$$L_n = C'(d, A, B) \frac{x_n}{\log^C x_n} T^{-\nu}$$

such that  $\Gamma_{x_n L_n}(\mathbb{Q}, T)$  is contained in one algebraic curve of degree  $\leq d$ .

**REMARK 2.16** For  $\gamma$  a slow parametrization of a curve  $\Gamma$ , with height control function  $\varphi$  and for  $T \geq 1$  fixed, since the sequence  $(x_n)_{n \in \mathbb{N}}$  goes to  $+\infty$ , we can cover the interval  $[N(C), \varphi(T)]$ , whose image contains the rational points of  $\Gamma$  of height  $\leq T$ , with a finite number of intervals  $[x_n, x_{n+1}]$ . An upper bound on this number provides an upper bound for the number of curves of degree  $\leq d$  containing  $\Gamma_{N(C), +\infty}(\mathbb{Q}, T)$ . The following Lemma gives such an upper bound.

**LEMMA 2.17** *Let  $\gamma$  be a slow parametrization of a plane curve  $\Gamma$ , with height control function  $\varphi$ , and let  $T \geq 1$  and  $d \in \mathbb{N}^*$ . We denote by  $n(T)$  the least  $n \in \mathbb{N}$  such that  $x_n \geq \varphi(T)$ . Then*

$$n(T) \leq \frac{T^\nu \log^{C+1} \varphi(T)}{\log(2) \min(1, C'(d, A, B))} + 1,$$

where  $C'(d, A, B)$  and  $\nu$  are given by Proposition 2.13. In particular, we can cover  $[N(C); \varphi(T)]$  with at most

$$\left\lceil \frac{T^\nu \log^{C+1} \varphi(T)}{\log(2) \min(1, C'(d, A, B))} \right\rceil + 1$$

intervals  $I$  such that  $\Gamma_1(\mathbb{Q}, T)$  is contained in one algebraic curve of  $\mathbb{R}^2$  of degree  $\leq d$ .

*Proof.* In case  $x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq \varphi(T)$ , we have

$$x_n \geq x_0 \left( 1 + C'(d, A, B) \frac{T^{-\nu}}{\log^C \varphi(T)} \right)^n.$$

Since by definition  $x_{n(T)-1} \leq \varphi(T)$ , we have

$$\varphi(T) \geq x_{n(T)-1} \geq x_0 \left( 1 + C'(d, A, B) \frac{T^{-\nu}}{\log^C \varphi(T)} \right)^{n(T)-1}.$$

In particular

$$n(T) \leq \frac{\log \varphi(T) - \log x_0}{\log(1 + C'(d, A, B) T^{-\nu} / \log^C \varphi(T))} + 1.$$

In case  $C'(d, A, B) T^{-\nu} / \log^C \varphi(T) \geq 1$ , one has

$$n(T) \leq \frac{\log \varphi(T)}{\log 2} + 1 \leq \frac{T^\nu \log^{C+1} \varphi(T)}{\log 2} + 1.$$

The last bound in this double inequality is harmless, since we will see in Theorem 2.20 that the term  $T^\nu$  is a constant for a good choice of  $d$  as a function of  $T$ . In case  $C'(d, A, B) T^{-\nu} / \log^C \varphi(T) \leq 1$ , by concavity of the log function, we obtain

$$n(T) \leq \frac{T^\nu \log^{C+1} \varphi(T)}{\log(2) C'(d, A, B)} + 1. \quad \square$$

**DEFINITION 2.18** We say that a curve  $\Gamma \subset \mathbb{R}^2$  is **transcendental** if it contains no infinite semi-algebraic set. A **Bézout bound for  $\Gamma$**  is any quantity  $\mathcal{B}(x, d)$  that dominates the number of points of  $\Gamma_{[d,x]} \cap P^{-1}(0)$  as  $P$  ranges over all non-zero polynomials in  $\mathbb{R}[X, Y]$  of degree at most  $d \in \mathbb{N}^*$ .

**NOTATION 2.19** For  $x \in \mathbb{R}$ , put  $\log_+ x = \max(1, \log x)$ .

**THEOREM 2.20** Let  $\gamma$  be a slow parametrization of a transcendental plane curve  $\Gamma$ , with height control function  $\varphi$ , and let  $T \geq 1$ . Then there exists a constant  $\alpha = \alpha(A, B, C)$  such that

$$\#\Gamma_{N(C), +\infty}(\mathbb{Q}, T) \leq \alpha \log_+^{2(B+C)}(T) \log^{C+1}(\varphi(T)) \mathcal{B}(\varphi(T), \log_+ T),$$

where  $N(C)$  is given by Proposition 2.12, and can for instance be  $e^C$ .

In particular,  $\Gamma$  has finite order as soon as  $\varphi$  is polynomially bounded and  $\mathcal{B}(x, d) \leq Q(\log x, d)$ , for some polynomial  $Q$ , or as soon as  $\varphi(T)$  is bounded by a power of  $\log(T)$  and  $\mathcal{B}(x, d) \leq Q(x, d)$ , for some polynomial  $Q$ .

*Proof.* By Lemma 2.17, the numbers of intervals  $I$  we need to cover  $[N(C), \varphi(T)]$ , in such a way that  $\Gamma_I(\mathbb{Q}, T)$  is contained in one algebraic curve of degree at most  $d$ , is less than

$$\frac{T^\nu \log^{C+1} \varphi(T)}{\log(2) \min(1, C'(d, A, B))} + 1,$$

with

$$\frac{1}{C'(d, A, B)} = A\mu^B \mu^{2d/\mu} (\mu!)^{2/(\mu(\mu-1))}.$$

Thus,  $\#I_{N(C,E),+\infty}(\mathbb{Q}, T)$  is bounded by

$$\left( \frac{T^\nu \log^{C+1} \varphi(T)}{\log(2) \min(1, C'(d, A, B))} + 1 \right) \mathcal{B}(\varphi(T), d).$$

By Proposition 2.13,  $\nu = \frac{4d}{\mu-1} \leq \frac{8d}{(d+1)(d+2)-2} \leq \frac{8}{d}$ . Taking  $d = \lfloor \log_+ T \rfloor$ ,

we find for instance  $T^\nu \leq e^{16}$ . Finally, let us bound each factor of  $1/C'(d, A, B)$ . We have, since  $\frac{d^2}{2} \leq \mu \leq 2d^2$ ,

$$\mu^B \leq 2^B (1 + \log_+ T)^{2B} \leq 2^B (1 + 1/\log 2)^{2B} \log_+^{2B} T.$$

$$\mu^{\frac{2d}{\mu}} = e^{\frac{4}{d} \log(2d^2)} \leq e^{12}.$$

$$(\mu!)^{\frac{2}{\mu(\mu-1)}} \leq \mu^{\frac{2}{\mu-1}} \leq e^{\frac{4 \log \mu}{\mu}} \leq e^{\frac{4}{e}}.$$

Now in case  $\min(1, C'(d, A, B)) = 1$ , we have that the number of intervals  $I$  we need to cover  $[N(C), \varphi(T)]$  is bounded by  $\alpha' \log^{C+1} \varphi(T)$ , for some real number  $\alpha'$ .

On the other hand, if  $\min(1, C'(d, A, B)) \neq 1$ , we have obtained as a bound  $\alpha'' \log_+^{2(B+C)} T \log^{C+1} \varphi(T)$ , for some real number  $\alpha''$  depending on our data  $A, B, C$ . In any case, one can take for a bound  $\alpha \log_+^{2(B+C)} T \log^{C+1} \varphi(T)$ , with  $\alpha = \max(\alpha', \alpha'')$ .

Assuming that  $\varphi$  is polynomially bounded and  $\mathcal{B}(x, d) \leq P(\log x, d)$ , for some polynomial  $P \in \mathbb{R}[X, Y]$ , one directly obtains that  $\Gamma_{N(C),+\infty}$  has finite order. But since  $\mathcal{B}(x, d) \leq P(\log x, d)$ , one also deduces that  $\Gamma_{[a, N(C)]}$  has finite order, by the same computation on  $[a, N(C)]$ , that reduces in the compact situation to the computation of [24] for mild parametrization.  $\square$

REMARK 2.21 The bound given for  $\#I_{N(C),+\infty}(\mathbb{Q}, T)$  in Theorem 2.20 is a product of two factors, one coming from the fast decay of the derivatives of the slow parametrization, and the other

one,  $\mathcal{B}(\varphi(T), \log_+ T)$ , depending on how much the curve intersects algebraic curves of degree  $\lfloor \log_+ T \rfloor$  on the parameter interval  $[N(C), \varphi(T)]$ . Those two factors may behave independently. We know, by [25, 30] or [31], of functions analytic on a neighbourhood of a compact interval having asymptotically as many as possible rational points of height less than  $T$  in their graph (for instance, for  $\epsilon \in ]0, 1[$ , more than  $\frac{1}{2} e^{2 \log^{1-\epsilon} T}$  points, for an infinite sequence of values of  $T$ , whereas it has to be less than  $C_\epsilon e^{\epsilon \log T}$  by [6] or [28]). Since the graph of a function analytic on a neighbourhood of a compact interval automatically inherits a mild parametrization, that is the compact version of our slow parametrization, it means that for those functions the second factor in our bound is as big as possible, while the first one stays bounded by a power of  $\log T$ . On the other hand, for some families of curves  $\Gamma$ , one knows that  $\Gamma$  cuts algebraic curves of given degree in few points with respect to this degree, and in this case, the split form of the bound of  $\# \Gamma_{N(C), +\infty}(\mathbb{Q}, T)$  in Theorem 2.20 provides a small value.

**REMARK 2.22** Parametrizations  $\gamma$  with stronger decay than the decay  $\varphi_p$  of smooth parametrizations of Definition 2.2 provide better bounds for  $\# \Gamma(\mathbb{Q}, T)$ . For instance, for some positive real number  $E$  and with the notation of Definition 2.2, one can consider parametrizations  $\gamma = (f, g)$  such that  $g$  is slow and  $f$  satisfies for all  $p \geq 0$

$$\left| \frac{(f - u)^{(p)}(x)}{p!} \right| \leq \frac{1}{x^E} \varphi_p(x). \quad (2.5)$$

In this case

- (1) By Remark 2.15(3), when  $u \in \mathbb{Q}$  one can take  $T^{\frac{K}{E}}$  as a height control function for  $\gamma$  if  $f - u$  has no zeros; by Remark 2.15(4), in case  $u \notin \mathbb{Q}$  has an irrationality measure function  $\tau(T)$  of the form  $K \log T$ ,  $K > 0$  (in particular when  $u$  is not a  $U$ -number of degree 1), then one can take  $T^{\frac{K}{E}}$  as height control function for  $\gamma$ .
- (2) Since the decay of the derivatives of  $f$  is improved by condition (2.5), comparing to the case where  $f$  is simply slow, assuming condition (2.5), the same computations lead in Lemma 2.10 to a denominator  $x^{p+E\alpha_1}$  instead of  $x^p$ , in Proposition 2.12 to a denominator  $N^{\rho+E\frac{\alpha_2}{2}}$  instead of  $N^\rho$ , and  $N(C, E) = e^{\frac{C}{E}}$  instead of  $N(C) = e^C$ . As a consequence, in Theorem 2.20, we obtain a smaller factor  $\log \varphi(T)$  instead of  $\log^{C+1} \varphi(T)$ .

### 3. Some examples

We apply in this section Theorem 2.20, the main statement of Section 2, which provides a bound for the number of rational points of prescribed height in a curve with slow parametrization  $\gamma$  and convenient height control function  $\varphi$  as soon as we know a convenient Bézout bound  $\mathcal{B}(x, d)$  for such a curve. In particular, we indicate how to obtain Proposition 1.1 by proving in detail a special case of Proposition 1.3, and similarly for Proposition 1.4.

#### 3.1. Spirals

One typical family of oscillatory curves that Theorem 2.20 allows us to treat is the following family of ‘fast’ logarithmic spirals. Let  $\ell, q \in \mathbb{N}^*$ ,  $F, G > 0$  and let

$$\phi_\ell := \frac{1}{x^F} \sin \circ \log^\ell(x), \quad \psi_q := \frac{1}{x^G} \cos \circ \log^q(x), \quad (3.1)$$

defined on some unbounded interval  $[a, +\infty[$  in  $\mathbb{R}_+^*$ , with  $a > 1$ . Put  $\gamma_{\ell,q} := (\phi_\ell, \psi_q)$ .

Observe that the image  $\Gamma$  of  $\gamma_{\ell,q}$  is the same as that of

$$t \mapsto (e^{-Ft} \sin^d t, e^{-Gt} \cos^d t), \quad t \geq \log a.$$

The following two Lemmas 3.1 and 3.2 will be used to prove in Lemma 3.3 that  $\gamma_{\ell,q}$  is a slow parametrization of the curve  $\Gamma$ .

LEMMA 3.1 For  $\ell \geq 1$ ,  $p \geq 1$  and  $x \geq e$ ,

$$\frac{(\log^\ell)^{(p)} x}{p!} \leq 2^\ell p^\ell \frac{\log^{\ell-1} x}{x^p}.$$

*Proof.* We use formula (2.2) with  $f = 1$ ,  $g = \log$  and  $\beta = \ell$ . We have

$$\frac{(\log^\ell)^{(p)} x}{p!} = \sum_{j_1 + \dots + j_\ell = p} \frac{g^{(j_1)}(x) \dots g^{(j_\ell)}(x)}{j_1! \dots j_\ell!}.$$

Let  $k$  be the number of non-zero indices  $j_r$  in the term  $\frac{g^{(j_1)}(x) \dots g^{(j_\ell)}(x)}{j_1! \dots j_\ell!}$ ; then this term is equal to

$$\frac{\log^{\ell-k} x}{x^p j_1 \dots j_\ell},$$

where only the non-zero  $j_r$ 's appear in the denominator. We then have, since  $\log x \geq 1$ ,

$$\frac{(\log^\ell)^{(p)} x}{p!} \leq (p+1)^\ell \frac{\log^{\ell-1} x}{x^p} \leq 2^\ell p^\ell \frac{\log^{\ell-1} x}{x^p}.$$

□

In the next Lemma, we use the following classical formula:

$$\frac{(f \circ g)^{(p)}(x)}{p!} = \sum_{1m_1 + 2m_2 + \dots + pm_p = p} \frac{f^{(m_1 + \dots + m_p)}(g(x)) \prod_{j=1}^p \left( \frac{g^{(j)}(x)}{j!} \right)^{m_j}}{m_1! \dots m_p!}. \quad (3.2)$$

LEMMA 3.2 Let  $d \in \mathbb{R}$  and  $f$  be a smooth function defined on  $[d, +\infty[$ . Let  $\alpha \geq 1$  be such that  $|f^{(p)}(x)| \leq \alpha^p$  for all  $x \geq d$  and  $p \geq 0$ :

- (1) If  $s \in S([e, +\infty[)$  with range in  $[d, +\infty[$ , then  $f \circ s \in S([e, +\infty[)$ .
- (2) If  $\ell \geq 1$ ,  $p \geq 0$  and  $x \geq e$ , then

$$\frac{|(f \circ \log^\ell)^{(p)}(x)|}{p!} \leq (\alpha 2^\ell)^p p^{(\ell+1)p} \frac{\log^{p(\ell-1)} x}{x^p}$$

(and so  $f \circ \log^\ell \in S([e, +\infty[))$ ).

*Proof.* (1) Let us denote by  $A$ ,  $B$  and  $C$  the constants attached to the slow function  $s$ . Note that  $f \circ s$  is bounded (by 1), hence so is  $f$ . By formula (3.2), one has for  $p \geq 1$

$$\begin{aligned} \left| \frac{(f \circ s)^{(p)}(x)}{p!} \right| &\leq \sum_{1m_1+2m_2+\dots+pm_p=p} \alpha^p \prod_{j=1}^p A^{jm_j} j^{Bjm_j} \frac{\log^{Cjm_j} x}{x^{jm_j}} \\ &\leq \alpha^p A^p p^{(B+1)p} \frac{\log^{Cp} x}{x^p}. \end{aligned}$$

(2) One cannot use direct statement (1), since strictly speaking,  $\log^\ell$  is not slow on  $[e, +\infty[$ , as it is not bounded, but nevertheless the computation made to prove (1) does not use that  $s$  is bounded and shows that, for any  $p \geq 1$ , one has the bound announced in (2). Indeed by Lemma 3.1, we can take the constants  $A = 2^\ell$ ,  $B = \ell$ ,  $C = \ell - 1$  for  $(\log^\ell)^{(p)}/p!$  to satisfy the bound of Definition 2.2, for any  $p \geq 1$ .  $\square$

**LEMMA 3.3** *Let  $d \in \mathbb{R}$ ,  $f$  and  $g$  be two smooth functions defined on  $[d, +\infty[$ , and  $F, G > 0$ . Assume that  $f$  and  $g$  are in the set of functions  $h$ , satisfying*

$$h = \text{Id or } \exists \alpha (\geq 1), \quad \forall x \geq d, \quad \forall p \geq 0, \quad |h^{(p)}(x)| \leq \alpha^p.$$

*Let  $s$  and  $\sigma$  be two slow functions on  $[e, +\infty[$  with range in  $[d, +\infty[$ . Then, for  $x \geq e$ , the parametrization*

$$x \mapsto \left( \frac{1}{x^F} f \circ s(x), \frac{1}{x^G} g \circ \sigma(x) \right)$$

*is a slow parametrization, satisfying condition (2.5) of Remark 2.22. In particular, the parametrization  $\gamma_{\ell,q} = (\phi_\ell, \psi_q)$  of (3.1) is a slow parametrization of the spiral  $\gamma_{\ell,q}([e, +\infty[)$ , satisfying condition (2.5), with height control function  $\varphi(T) = T^{\frac{1}{\min(F,G)}}$ .*

*Proof.* The functions  $h(x) = \frac{1}{x^F} f \circ s(x)$  and  $k(x) = \frac{1}{x^G} g \circ \sigma(x)$  are slow since the functions  $\frac{1}{x^F}$  and  $\frac{1}{x^G}$  are slow, as well as  $f \circ s$  and  $g \circ \sigma$ , by Lemma 3.2. It is immediate that  $h(x)$  and  $k(x)$  satisfy condition (2.5) of Remark 2.22. Nevertheless, the following computation is provided in order to make explicit the constants attached to the slowness of  $(h, k)$ . We have by formula (2.2),

$$\frac{h^{(p)}(x)}{p!} = \sum_{i=0}^p \frac{1}{(p-i)!} \left( \frac{1}{x^F} \right)^{(p-i)} \frac{1}{i!} (f \circ s)^{(i)}(x).$$

Observe that

$$\frac{1}{(p-i)!} \left( \frac{1}{x^F} \right)^{(p-i)} \leq \frac{F}{1} \frac{F+1}{2} \cdots \frac{F+p-i-1}{p-i} \frac{1}{x^{F+p-i}} \leq \frac{1}{x^F} \frac{(F+1)^{p-i}}{x^{p-i}}.$$

It follows, by Lemma 3.2, that for any  $x \geq e$ , denoting again  $A, B, C$  the constants attached to the slow function  $s$ ,

$$\begin{aligned} \left| \frac{h^{(p)}(x)}{p!} \right| &\leq \sum_{i=0}^p \frac{1}{x^F} \frac{(F+1)^{p-i}}{x^{p-i}} \alpha^i A^i i^{(B+1)i} \frac{\log^{iC} x}{x^i} \\ &\leq \frac{1}{x^F} (p+1) (F+1)^p \alpha^p A^p p^{(B+1)p} \frac{\log^{pC} x}{x^p}. \end{aligned}$$

Since for  $p \geq 1$ , one has for instance  $p+1 \leq 2^p$ , one finally obtains

$$\left| \frac{h^{(p)}(x)}{p!} \right| \leq \frac{1}{x^F} \left[ 2(F+1) \alpha A p^{B+1} \frac{\log^C x}{x} \right]^p,$$

and in the same way, denoting  $a, b$  and  $c$ , the constants attached to the slow function  $\sigma$ ,

$$\left| \frac{k^{(p)}(x)}{p!} \right| \leq \frac{1}{x^G} \left[ 2(G+1) \alpha a p^{b+1} \frac{\log^c x}{x} \right]^p,$$

showing that  $(h, k)$  satisfies condition (2.5) of Remark 2.22 for the following set of four constants

$$2\alpha (\max\{F, G\} + 1) \max\{A, a\}, \max\{B, b\} + 1, \max\{C, c\} \quad \text{and} \quad E = \max\{F, G\}.$$

In particular, by Lemma 3.2,  $(\phi_\ell, \psi_q)$  satisfies condition (2.5) of Remark 2.22 for the constants:

$$2\alpha (\max\{F, G\} + 1) \max\{\ell, q\}, \max\{\ell, q\} + 1, \max\{\ell, q\} - 1 \quad \text{and} \quad E = \max\{F, G\}.$$

By Remark 2.15(2), one can take  $\varphi(T) = T^{\frac{1}{\min\{F, G\}}}$  for a height control function.  $\square$

We now give an explicit possible value for the bound  $\mathcal{B}(x, d)$  (defined in Theorem 2.20) relative to the slow parametrized spiral of (3.1).

We are searching for a bound  $\mathcal{B}(L, d)$  for the number of solutions of

$$P \left( \frac{1}{x^F} \sin \circ \log^\ell x, \frac{1}{x^G} \cos \circ \log^q x \right),$$

for  $x$  in some subinterval of  $[1, +\infty[$  of length less than  $L$ , and for  $P \in \mathbb{R}[X, Y]$  of degree less than  $d$ . This amounts to bounding the number of solutions of

$$Q(x, \sin \circ \log^\ell(x), \cos \circ \log^q(x)) = 0,$$

for  $x$  in some subinterval of  $[1, +\infty[$  of length less than  $L$ , and for  $Q \in \mathbb{R}[X, Y, Z]$  of degree less than  $d(F+G)$ . This finally amounts to bounding the number of solutions of the system

$$Q(e^y, \sin(z), \cos(w)) = z - y^\ell = w - y^q = 0,$$

for  $y$  in some subinterval of  $[1, +\infty[$  of length less than  $\log L$ , and for  $Q \in \mathbb{R}[X, Y, Z]$  of degree less than  $d(F + G)$ . By the theorem in [21, Section 1.4], we obtain

$$\mathcal{B}(L, d) \leq 4d(F + G)\ell q(d(F + G) + \ell + q + 2)^2(\lfloor (\log L)/\pi \rfloor + 1).$$

Note that by Gwoździewicz *et al.* [17, Lemma 3] or Benedetti and Risler [2, Lemma 4.2.6], we can dispose of the non-degeneracy hypothesis in the theorem of [21, Section 1.4] by bounding instead the number of solutions of the regular system

$$Q(e^y, \sin(z), \cos(w)) = \epsilon, z - y^\ell = w - y^q = 0,$$

for  $\epsilon$  a regular value of  $y \mapsto Q(e^y, \sin(y^\ell), \cos(y^q))$ .

For an appropriate constant  $\alpha'$ , using the bound  $\alpha' \log_+^4 T$  we just obtained for  $\mathcal{B}\left(T \frac{1}{\min(F, G)}, \log T\right)$  and using the constants attached to the slow parametrization (3.1) that we obtained at the end of the proof of Lemma 3.3, one sees by Theorem 2.20 and Remark 2.22 that for  $\Gamma$  the spiral parametrized by (3.1), one can state the following proposition.

**PROPOSITION 3.4** *Let  $F, G > 0$ ,  $\ell, q \in \mathbb{N}^*$  and  $T \geq 1$ . Then there exist  $N = N(F, G, \ell, q)$  (we can take  $N = e^{\frac{\max\{\ell, q\}-1}{\max\{F, G\}}}$ ) and constants  $\alpha = \alpha(F, G, \ell, q)$  and  $\beta$  (we can take  $\beta = 5 + 4 \max\{\ell, q\}$ ) such that*

$$\#\Gamma_{N, +\infty}(\mathbb{Q}, T) \leq \alpha \log_+^\beta T,$$

for  $\Gamma$  the spiral parametrized by  $\gamma_{\ell, q}(x) = \left(\frac{1}{x^F} \sin \circ \log^\ell(x), \frac{1}{x^G} \cos \circ \log^q(x)\right)$ .

**REMARK 3.5** In Proposition 3.4, we could replace the spiral parametrized by  $\gamma_{\ell, q}$  by a transcendental curve  $\Gamma$  parametrized by

$$x \mapsto \left(u + \frac{1}{x^F} f \circ s(x), v + \frac{1}{x^G} g \circ \sigma(x)\right),$$

where  $F, G > 0$ ,  $u, v \in \mathbb{R}$  and

(1)  $f$  and  $g$  are in the set of functions  $h$ , satisfying

$$h = \text{Id or } \exists \alpha (\geq 1), \quad \forall x \geq d, \quad \forall p \geq 0, \quad |h^{(p)}(x)| \leq \alpha^p,$$

(2)  $f, g, s$  and  $\sigma$  are elementary functions in the sense of [21, Section 1.5] (defined from the simple functions  $e^x, \sin x, \cos x, \log x, \arcsin x, \arccos x, \tan x, \arctan x$  and rational functions, by induction using composition),

(3)  $s$  and  $\sigma$  are compositions of slow functions, respectively, with  $\log^\ell$  and  $\log^q$ , for some  $\ell, q \in \mathbb{N}^*$ ,

(4) one of the following conditions on  $u$  and  $v$  is satisfied:

- $u \in \mathbb{Q}$  and  $f$  has no zeros,
- $v \in \mathbb{Q}$  and  $g$  has no zeros,
- $u$  and  $v$  are both rational,
- $u \notin \mathbb{Q}$  and  $u$  is not a  $U$ -number of degree  $\nu = 1$  in Mahler's classification,
- $v \notin \mathbb{Q}$  and  $v$  is not a  $U$ -number of degree  $\nu = 1$  in Mahler's classification.

In this situation, on one hand, this parametrization is slow by Lemma 3.3, and on the other hand, by the theorem of [21, Section 1.6],  $\mathcal{B}(L, d)$  is polynomially bounded in  $d$  and  $\log_+ L$ . Furthermore, by Remarks 2.15, any of the conditions (1), (2), (3) or (4) on  $u$  and  $v$  ensure that one can take a power of  $T$  as a height control function for  $\gamma$ . (In order to apply Remark 2.15(2) to condition (3), observe that every common zero of  $f \circ s$  and  $g \circ \sigma$  maps to the single point  $(u, v)$ .) In conclusion, our assumptions on  $f, g, s, \sigma, u$  and  $v$  imply, in the same way as for Proposition 3.4, that  $\Gamma$  has finite order.

Note that functions  $f, g, s$  and  $\sigma$  satisfying the above conditions can be built using Remarks 2.6. For instance,  $f$  and  $g$  can be built from  $\sin, \cos, \arctan$  and rational functions of negative degree (which are bounded elementary functions, and composition of those functions with elementary functions), in order to get elementary bounded functions. Functions in the algebra generated by bounded rational functions in  $\log^r, \sin \circ \log^\ell, \cos \circ \log^g$  and  $\arctan \circ \log^m$  are instances of slow functions  $s$  and  $\sigma$ .

*Proof of Propositions 1.1 and 1.3.* The proof of Propositions 1.1 and 1.3 is straightforward, after reparametrization by  $\log$ , since the assumptions of Proposition 1.1 are an axiomatization of the proof of Proposition 3.4, as made in Remark 3.5. Thus, Remark 3.5 provides a compact interval  $J$  of the parameter outside which the curve of Proposition 1.1 has finite order. But for the piece of this curve parametrized by  $J$ , one can apply again our computation, which reduces in the compact case exactly to the computation of [24] through mild parametrizations.  $\square$

### 3.2. The case of graphs

We proceed here similarly as in the preceding section to establish Proposition 1.4 as a corollary of a detailed proof of a special case.

Given a function  $g: J \rightarrow \mathbb{R}$  on some interval  $J$  of  $\mathbb{R}$ , we denote by  $\Gamma$  (or if needed  $\Gamma_g$ ) the graph of  $g$ , and for  $I \subset J$ , we denote by  $\Gamma_I$  (or  $\Gamma_{g,I}$ ) the set  $\Gamma \cap (I \times \mathbb{R})$ .

We begin by noting that the case of  $\Gamma$  (for  $g$  having controlled decay) is encompassed by the discussion of Section 2. Indeed, for  $g: [1, +\infty[ \rightarrow \mathbb{R}$  satisfying the conditions of Definition 2.2, the map  $\gamma: [1, +\infty[ \rightarrow \mathbb{R}^2$  defined by  $\gamma(x) = \left(\frac{1}{x}, g(x)\right)$  is a slow parametrization of a curve of  $\mathbb{R}^2$ , with rational points in bijective correspondence with rational points of same height in  $\Gamma$ . Moreover,  $\gamma$  satisfies condition (2.5) of Remark 2.22, and so as a consequence of Remark 2.22 and Theorem 2.20, we can state the following proposition.

**PROPOSITION 3.6** *Let  $g: [a, +\infty[ \rightarrow \mathbb{R}$  be a slow function with constants  $A, B$  and  $C$ . Assume that there exists some function  $\varphi: [1, +\infty[ \rightarrow \mathbb{R}$  such that the height of any rational point of  $\Gamma_{\varphi(T), +\infty}$  is  $\geq T$ . Then there exist  $N = N(C)$  (given by Proposition 2.12 and that can be for instance  $e^C$ ) and a constant  $\alpha = \alpha(A, B, C)$  such that*

$$\# \Gamma_{N,+\infty}(\mathbb{Q}, T) \leq \alpha \log_+^{2(B+C)} T \log(\varphi(T)) \mathcal{B}(\varphi(T), \log_+ T).$$

In particular, since one can always take  $\varphi(T) = T$  by Remark 2.15(3),

$$\# \Gamma_{N,+\infty}(\mathbb{Q}, T) \leq \alpha \log_+^{2(B+C)+1} T \mathcal{B}(T, \log T).$$

Hence,  $\Gamma$  has finite order as soon as there exists a polynomial  $Q$  such that  $\mathcal{B}(x, d) \leq Q(\log x, d)$ .

**REMARK 3.7** By the Lindemann–Weierstrass theorem, there are no rational points in the graph of  $\sin$  (resp.  $\log$ ) except the point  $(0, 0)$  (resp.  $(1, 0)$ ), since for a non-zero algebraic number  $x \in \mathbb{C}$ ,  $e^x$  is transcendental. A natural way to build functions from  $\sin$  with graph having *a priori*, the most chance to contain rational points is to compose  $\sin$  with a function sending rational points to transcendental ones, such as  $x \mapsto rx$ , for  $r$  a transcendental number or  $x \mapsto \log^\ell x$ , for  $\ell \in \mathbb{N}^*$ . Proposition 3.6 allows us to treat both cases.

In the first case, for the function  $x \mapsto \sin(cx)$ ,  $c \in \mathbb{R}_+^*$ , restricted to a compact interval  $\left[ a - \frac{\pi}{2c}, a + \frac{\pi}{2c} \right]$ , where  $a \in \mathbb{R}$ , the method of proofs of Section 2 applies, and reduces to the methods of [6, 24] for analytic functions defined on compact intervals. In this situation, since [21, Section 1.4] provides  $\alpha' \log^2 T$  as bound for the number of solutions of  $P(x, \sin(cx)) = 0$ ,  $x \in \left[ a - \frac{\pi}{2c}, a + \frac{\pi}{2c} \right]$ ,  $\deg(P) = \lfloor \log_+ T \rfloor$ , we get a bound  $\# \Gamma_{\left[ a - \frac{\pi}{2c}, a + \frac{\pi}{2c} \right]} \leq \alpha'' \log_+^2 T$ . Here, the constant  $\alpha''$  does not depend on  $a$ , since we have uniform bounds for the derivatives of  $x \mapsto \sin(cx)$  with respect to  $a$ . Consequently for the graph  $\Gamma_c$  of  $x \mapsto \sin(cx)$ , we have

$$\# \Gamma_{c,\mathbb{R}}(\mathbb{Q}, T) \leq \alpha c T \log_+^2 T.$$

This bound is quite sharp, since  $\# \Gamma_{\frac{1}{n},\mathbb{R}}(\mathbb{Q}, T)$  is bounded from below by  $\frac{\alpha''}{n} T$  for  $T \geq n$ .

The second case cannot be reduced to the compact case and uniform bounds for derivatives with respect to translations, and thus requires control on the derivatives at infinity, as in the assumption of Proposition 3.6. We hereafter treat this case as a consequence of Proposition 3.6.

**COROLLARY 3.8** Let  $\ell \in \mathbb{N}^*$ ,  $g_\ell: \mathbb{R}_+^* \rightarrow \mathbb{R}$  be the function defined by  $g_\ell(x) = \sin \circ \log^\ell(x)$  and let  $\Gamma_\ell$  its graph. Then there exist constants  $\alpha = \alpha_\ell$  and  $\beta = \beta_\ell$  ( $\beta = 5 + 4\ell$  being possible) such that for any  $T \geq 1$ ,

$$\Gamma_\ell(\mathbb{Q}, T) \leq \alpha \log_+^\beta T.$$

*Proof.* Using the theorem of [21, Section 1.4] in the same way that we did for Proposition 3.4, one obtains here, for the curve  $\Gamma_\ell \cap ([1, T] \times \mathbb{R})$ , the bound  $\mathcal{B}(T, d) \leq 4d\ell(d + \ell + 2)^2 \left( \left\lfloor \frac{\log T}{\pi} \right\rfloor + 1 \right)$ . Since by Lemma 3.3, the derivatives of  $g_\ell$  satisfy the bound required by Proposition 3.6, one deduces from this proposition the existence of constants  $N$ ,  $\alpha$  and  $\beta$ , depending only on  $\ell$ , such that

$$\# \Gamma_{\ell,[N,+\infty]}(\mathbb{Q}, T) \leq \alpha \log_+^\beta T.$$

Since  $\mathcal{B}(T, \log T) \leq \alpha'_\ell \log^4 T$ , for some  $\alpha'_\ell > 0$ , by Lemma 3.2 and by Proposition 3.6,  $\beta = 5 + 4\ell$  is possible. Assuming that  $N \geq 1$ , a bound on the same kind also holds over the interval  $]0, 1/N[$ , since the one-to-one transformation  $(x, y) \mapsto (1/x, -y)$  maps the rational points of height less than  $T$  of  $\Gamma_{\ell, ]0, 1/N[}$  onto the rational points of height at most  $T$  of  $\Gamma_{\ell, [N, +\infty[}$ . Finally, over  $[1/N, N]$ , by [24], we also have the same kind of bound for  $\#\Gamma_{\ell, [1/N, N]}(\mathbb{Q}, T)$ , since  $g_{\ell, [1/N, N]}$  is an analytic function on a compact domain, and thus this graph comes with its obvious mild-parametrization (see the discussion after Theorem 1.5 in [24]).  $\square$

REMARK 3.9 More generally, and similar to Remark 3.5 and the proofs of Propositions 1.1 and 1.3, one can consider transcendental elementary functions defined in [21] that are given by composition of a slow function with some power of  $\log$  (to have a Bézout bound  $\mathcal{B}(x, d)$  as required in Proposition 3.6), in order to get instances of graphs with finite order. For instance, the graph of any coordinate of the curve of Example 1.2 has finite order. In this way, we get in particular a proof of Proposition 1.4. Concerning Proposition 1.4 as stated in the introduction, note that the function  $af(c \log^\ell)$  might not be bounded by 1 in absolute value (as required in the definition of slow function), but then  $af(c \log^\ell)/|a+1|$  is slow, and finally if  $\alpha \log^\beta T$  bounds  $\# \Gamma_{\frac{af(c \log^\ell)}{|a+1|}}(\mathbb{Q}, T)$  for some  $\alpha, \beta$ , then  $\alpha' \log^\beta T$  bounds  $\# \Gamma_{af(c \log^\ell)}(\mathbb{Q}, T)$ , for some  $\alpha'$ .

REMARK 3.10 Similar results stating that the graph of some function has finite order have been recently proved, in particular for entire functions from  $\mathbb{C}$  to  $\mathbb{C}$  (see [7]), where Bézout estimates are provided by the growth of these functions in the spirit of Coman and Poletsky [12]. In the real case, since the growth of the derivatives is not prescribed by the growth of the function itself, one has to consider some bound for all the derivatives as an assumption. For instances of functions with graphs of finite order (over a compact interval), see [13], where indeed the assumptions concern the Taylor coefficients at some point of the series.

REMARK 3.11 When it comes to counting rational points on graphs, a classical function to look at is the Riemann zeta function  $\zeta: ]1, +\infty[ \rightarrow \mathbb{R}$ , given by  $\zeta(x) = \sum_{n=1}^{+\infty} \frac{1}{n^x}$ . Let us denote its graph by  $\Gamma_\zeta$ . By van den Dries and Speissegger [15],  $\Gamma_\zeta$  is o-minimal, so  $\#\Gamma_\zeta(\mathbb{Q}, T)$  is sub-polynomial by [28]. Moreover, it is known since [22] that for some constant  $\alpha > 0$

$$\#\Gamma_{\zeta, ]2, 3[}(\mathbb{Q}, T) \leq \alpha \frac{\log^2 T}{(\log \log T)^2}.$$

The interval  $]2, 3[$  may be replaced by any bounded interval, as proved in [3]. In [8], it is finally proved that one can bound  $\#\Gamma_{\zeta, ]1, +\infty[}(\mathbb{Q}, T)$  (as well as the number of algebraic points of height  $\leq T$  and degree  $\leq k$  over  $\mathbb{Q}$ ) in the following way: for some constant  $\alpha > 0$ ,

$$\#\Gamma_{\zeta, ]1, +\infty[}(\mathbb{Q}, T) \leq \alpha \log^3(T) \log^3 \log T.$$

It is indicated in [8, page 1154] that one can even get a  $\log^2(T) \log^2 \log T$  bound. We can here easily give a bound in the form  $\log^4(T) \log \log T$  as a consequence of Proposition 3.6.

Another classical special function that can be treated by our approach is the Euler  $\Gamma$  function defined by  $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ , considered for  $x \geq 1$ . Let  $\Gamma_\Gamma$  be the graph of this function. As

for the Riemann zeta function,  $\Gamma$  is o-minimal (again see [15]). It has been proved in [4] that for any interval  $I$  of length 1,

$$\#\Gamma_{I,I}(\mathbb{Q}, T) \leq \alpha \frac{\log^2 T}{\log \log T},$$

and in [8] the following bound is given:

$$\#\Gamma_{I,]0,+\infty[} \leq \alpha \log^2(T) \log^3 \log T.$$

We give below a very rough bound of the form  $\log^{11}(T) \log \log T$  as a direct application of Proposition 3.6.

For the special functions  $\zeta$  and  $\Gamma$ , our bounds have no better exponents than the best known exponents. But be aware that no rational points are expected, or at least known, in the graphs of these functions, except for  $(n, n!)$ ,  $n \in \mathbb{N}^*$ , for  $\Gamma$ . Thus, even a small bound of type  $\alpha \log^\beta T$  is probably very far from being optimal. Note that by [12, Theorem 8.2], the Bézout estimate that we shall use for  $\zeta$  is quite sharp; this just shows that any existing general method based on Bézout estimate probably will not produce particularly sharp bounds. We produce bounds here only to show that our method works for these particular instances of functions, viewed as real ones, and how it can be adapted for the Euler function, since this function is not slow. Toward this end, we try to present shorter computations that are probably not the sharpest that one can get.

*The Riemann  $\zeta$  function.* First of all, we observe that the derivatives of  $\zeta$  satisfy the bound of Proposition 3.6 on any interval  $[a, +\infty[ \subset ]1, +\infty[$ , for some  $A = A_a, B = 1$  and  $C = 0$ . Indeed, we have on one hand for  $p \geq 0$ ,  $\zeta^{(p)} = \sum_{n \geq 1} (-1)^p \frac{\log^p n}{n^x}$ . On the other hand, since the study of  $x \mapsto \frac{x^p}{n^x}$  shows that  $\frac{1}{n^x} \leq \frac{p^p}{e^p x^p \log^p n}$ , for any  $a > 1$ , one can choose  $\lambda = \frac{1}{2} - \frac{1}{2a} \in ]0, 1[$ , such that for any  $x \geq a$  and  $p \geq 1$ ,

$$\begin{aligned} |\zeta^{(p)}(x)| &\leq \sum_{n \geq 1} \frac{\log^p n}{n^{\lambda x}} \frac{1}{n^{(1-\lambda)x}} \leq \frac{p^p}{(\lambda e)^p x^p} \zeta((1-\lambda)x) \\ &\leq \frac{p^p}{(\lambda e)^p x^p} \zeta\left(\frac{a}{2} + \frac{1}{2}\right) \leq \left(\frac{\zeta\left(\frac{a}{2} + \frac{1}{2}\right)}{\lambda e}\right)^p \frac{p^p}{x^p}. \end{aligned}$$

Then observe that, with the same  $\lambda$  as above and the notation  $m_a = \zeta\left(\frac{a}{2} + \frac{1}{2}\right) - 1$ , for  $\zeta$  one can take for a height function control

$$\varphi(T) = \frac{\log(m_a T)}{\lambda \log 2}. \tag{3.3}$$

This follows from the following remarks. In case  $\zeta(x) \in \mathbb{Q}$  as height less than  $T$ ,  $T \geq 1$ , then  $\zeta(x) - 1 \geq \frac{1}{T}$ . But for any  $x \in [a, +\infty[$ , one has

$$\zeta(x) - 1 \leq \sum_{n \geq 2} \frac{1}{n^{\lambda x}} \frac{1}{n^{(1-\lambda)x}} \leq m_a 2^{-\lambda x}.$$

Consequently, for  $x \geq \frac{\log(m_a T)}{\lambda \log 2}$ ,  $\zeta(x)$  has height at least  $T$ .

Finally, by [12, Theorem 8.2] or [22, Proposition 1], one knows that for some constant  $c > 0$ ,

$$\mathcal{B}(\varphi(T), \log T) \leq c(\log(T) + \varphi(T) \log \varphi(T)) \log T. \quad (3.4)$$

From (3.3), (3.4) and Proposition 3.6, we deduce that for any  $a > 1$  and  $T \geq 3$ ,

$$\Gamma_{\zeta, [a, +\infty[}(\mathbb{Q}, T) \leq \alpha \log^4(T) \log \log T.$$

**REMARK 3.12** For  $a > 1$ , on  $[a, +\infty[$ ,  $\zeta$  may be not bounded by 1, and thus on this interval  $\zeta$  is not a slow function. But as already noticed in Remark 2.3, one can always divide  $\zeta$  by some large enough integer  $M_a$  in order to fulfil the definition of slow function, since  $\alpha' \log^\beta T$  bounds  $\# \Gamma_{\zeta, [a, +\infty[}(\mathbb{Q}, T)$  whenever  $\alpha \log^\beta T$  bounds  $\# \Gamma_{\frac{\zeta}{M_a}, [a, +\infty[}(\mathbb{Q}, T)$ .

**REMARK 3.13** For  $u, v$  two real numbers such that  $u + v$  is irrational and not a  $U$ -number of degree 1, one deduces from the study above that the graph of  $u + v\zeta$  has finite order on  $[a, +\infty[$ . Indeed, up to dividing by a large enough integer as observed in Remark 3.12, one has that  $u + v\zeta$  is slow. Furthermore, a Bézout bound for  $\zeta$  is also a Bézout bound for  $u + v\zeta$ . Finally, taking into account the form of the Bézout bound for  $\zeta$  given in (3.4), to apply Proposition 3.6, it remains to prove that some power of  $\log T$  is a height control function for the graph of  $u + v\zeta$ . For this, observe that on one hand

$$|u + v\zeta(x) - (u + v)| \leq |v|m_a 2^{-\lambda x},$$

and on the other hand, since an irrationality measure function for  $u + v$  is of the form  $K \log T$  ( $K > 0$ ), whenever  $u + v\zeta(x)$  is a rational number of height  $\leq T$ , one has

$$\frac{1}{T^K} \leq |u + v\zeta(x) - (u + v)|.$$

It follows that if  $u + v\zeta(x)$  is a rational number of height  $\leq T$ , then  $\frac{1}{T^K} \leq |v|m_a 2^{-\lambda x}$ , and so  $x \leq K' \log T$  for some  $K' > 0$ .

*The Euler  $\Gamma$  function.* We first remark that since for any  $p \geq 0$ , since  $\frac{\log^p t}{t} \leq \left(\frac{p}{e}\right)^p$  and since  $\Gamma^{(p)}(x) = \int_0^{+\infty} \log^p(t) t^{x-1} e^{-t} dt$ , one has

$$\Gamma^{(p)}(x) \leq \left(\frac{p}{e}\right)^p \Gamma(x+1) = \left(\frac{p}{e}\right)^p x \Gamma(x). \quad (3.5)$$

Now let us denote by  $f$  the inverse function of  $\Gamma$  on  $[1, +\infty[$ , and  $x = f(y)$ . One can show by induction on  $p \geq 1$  that  $\frac{f^{(p)}(y)}{p!}$  is a sum of at most  $p!$  terms of the form

$$c(\Gamma^{(j_1)}(x))^{m_1} \dots (\Gamma^{(j_p)}(x))^{m_p} (\Gamma'(x))^{-k},$$

with  $|c| \leq 2^{p-2}$ ,  $k \in [0, 2p - 1]$ ,  $j_1, \dots, j_p \in [2, p]$ ,  $m_1 + \dots + m_p \in [0, p - 1]$ ,  $-k + j_1 m_1 + \dots + j_p m_p = -1$  and  $-k + m_1 + \dots + m_p = -p$ . From this observation and from (3.5), one has for any  $p \geq 1$  and any  $y \geq 1$  and for a set  $J$  of indices  $m_i, j_r$  of cardinality less than  $p!$ ,

$$\begin{aligned} \frac{f^{(p)}(y)}{p!} &\leq 2^p \sum_J \left(\frac{p}{e}\right)^{\sum_{r=1}^p j_r m_r} (x\Gamma(x))^{\sum_{r=1}^p m_r} (\Gamma'(x))^{-k} \\ &\leq \left(\frac{2}{e^2}\right)^p p^{2p} \sum_J (x\Gamma(x))^{\sum_{r=1}^p m_r} (\Gamma(x))^{-k} \\ &\leq p! \left(\frac{2}{e^2}\right)^p p^{2p} \left(\frac{x}{\Gamma(x)}\right)^p = \left(\frac{2}{e^2}\right)^p p^{3p} \left(\frac{f(y)}{y}\right)^p. \end{aligned}$$

Since for some constant  $D$ , for any  $x \geq 1$ ,  $\Gamma(x) \geq De^x$ , we have for some constant  $\delta > 0$ , for any  $y \geq 1$ ,  $\log y \geq \delta f(y)$ , and thus one has for any  $y, p \geq 1$

$$\left| \frac{f^{(p)}(y)}{p!} \right| \leq \left(\frac{2}{\delta e^2}\right)^p p^{3p} \left(\frac{\log y}{y}\right)^p. \tag{3.6}$$

This does not show that  $f$  is slow, since  $f$  is not bounded. But, as already noted in Remark 2.5, a direct computation using  $f(y) \geq 1$ , formula (3.2) and the inequality (3.6) shows that  $1/f$  is slow, with constants  $A = \frac{4}{\delta e^2}$ ,  $B = 4$ ,  $C = 1$ . Note that

$$\begin{aligned} \#\Gamma_{\frac{1}{f}, [1, T]}(\mathbb{Q}, T) &= \#\Gamma_{f, [1, T]}(\mathbb{Q}, T) \\ &= \#\Gamma_{\Gamma, [1, \Gamma^{-1}(T)]}(\mathbb{Q}, T) \\ &= \#\Gamma_{\Gamma, [1, +\infty[}(\mathbb{Q}, T). \end{aligned}$$

In order to bound  $\#\Gamma_{\frac{1}{f}, [1, T]}(\mathbb{Q}, T)$  using Proposition 3.6, we need to produce for  $1/f$  a bound  $b_{1/f}(T, \log T)$ . But clearly it is enough to find a bound  $b_{\Gamma}\left(\frac{\log T}{\delta}, \log T\right)$  for  $\Gamma$ , since again,  $\log y \geq \delta f(y)$ . Such a bound is provided by [4, Proposition 3.1] in the form

$$b_{\Gamma}\left(\frac{\log T}{\delta}, \log T\right) \leq c \log^2(T) \log \log T.$$

We conclude by Proposition 3.6 that

$$\#\Gamma_{\Gamma, [1, +\infty[}(\mathbb{Q}, T) \leq \alpha \log^{11}(T) \log \log T.$$

#### 4. Some connections to logic

In this final section, the reader is assumed to be familiar with definability theory over the field of real numbers (see, for example, van den Dries and Miller [14] or Wilkie [33] for a brief introduction).

Given  $E \subseteq \mathbb{R}^n$ , we let  $E^{\text{trans}}$  be the result of removing from  $E$  all infinite and connected semialgebraic subsets. (Hence, the only non-empty semialgebraic subsets of  $E^{\text{trans}}$  are singletons.)

Let  $\mathfrak{R}$  be a fixed, but arbitrary, structure on the real field; ‘definable’ means ‘definable with parameters’, unless indicated otherwise.

The seminal paper [28] established the possibility of obtaining uniform large-scale asymptotics on height bounds of definable sets. We recall the basic result:

**THEOREM 4.1** [28, 1.8] *If  $\mathfrak{R}$  is o-minimal and  $E$  is definable, then  $\#E^{\text{trans}}(\mathbb{Q}, T)$  is sub-polynomial.*

This is the best possible bound in this generality (see [25, 30, 31] for information). Two questions arise naturally:

- (A) To what extent is o-minimality necessary?
- (B) Are there examples of  $\mathfrak{R}$  such that  $E^{\text{trans}}$  has finite order for every definable set  $E$ ?

There is a trivial positive answer to (B), because  $E^{\text{trans}}$  is finite if and only if  $E$  is semialgebraic, if and only if  $E$  is definable in the real field. Thus, we modify the question:

- (B') Are there examples of  $\mathfrak{R}$  such that  $E^{\text{trans}}$  has finite order for every definable set  $E$ , and there is some definable  $S$  such that  $S^{\text{trans}}$  contains a compact set of positive topological dimension?

By Binyamini and Novikov [5],  $\mathbb{R}^{\text{RE}}$  (the expansion of the real field by the restrictions of  $\exp$  and  $\sin$  to  $[0, 1]$ ) provides a positive answer to (B'). It is well known by now that  $\mathbb{R}^{\text{RE}}$  is o-minimal. Wilkie has conjectured that the expansion of the real field by  $\exp$  (on all of  $\mathbb{R}$ ) is another example; for more information on Wilkie's Conjecture and progress theretoward, see [5, 11, 19, 20, 24, 26, 27]. We have a conjecture of our own:

**CONJECTURE**  $E^{\text{trans}}$  has finite order for each  $E$  definable in the expansion of  $\mathbb{R}^{\text{RE}}$  by any logarithmic spiral  $\mathbb{S}_\omega$ . (We regard our result that  $\mathbb{S}_\omega$  has finite order as an encouraging first step.) A strictly (by Tychonievich [32]) weaker version:  $E^{\text{trans}}$  has finite order for each  $E$  definable in  $(\mathbb{R}, +, \cdot, \mathbb{S}_\omega)$ . Even weaker (potentially):  $E^{\text{trans}}$  has finite order for each  $E$   $\emptyset$ -definable in  $(\mathbb{R}, +, \cdot, \mathbb{S}_\omega)$ .

As for question (A), there is an obvious necessary condition: if  $\mathfrak{R}$  defines the set of integers,  $\mathbb{Z}$ , then we cannot have even  $\#E^{\text{trans}}(\mathbb{Q}, T) = o(T)$  for every  $\emptyset$ -definable  $E \subseteq \mathbb{R}$ . We can do better:

**PROPOSITION 4.2** *The following are equivalent:*

- (1)  $\#E^{\text{trans}}(\mathbb{Q}, T) = o(T)$  for every definable  $E \subseteq \mathbb{R}$ .
- (2) Every definable subset of  $\mathbb{R}$  either has interior or is nowhere dense.
- (3) For every definable  $E \subseteq \mathbb{R}^n$ , if no coordinate projection of  $E$  has interior, then  $\#E(\mathbb{Q}, T)$  is sub-polynomial.

(And similarly with ‘definable’ replaced by ‘ $\emptyset$ -definable’.)

*Proof.* 1  $\Rightarrow$  2. Let  $E \subseteq \mathbb{R}$  be definable and have no interior. Suppose to the contrary that  $E$  is dense in some non-empty open interval  $I$ ; then  $(I \cap E)^{\text{trans}} = I \cap E$  and  $(I \setminus E)^{\text{trans}} = I \setminus E$ , yielding  $\#I(\mathbb{Q}, T) = \#(I \cap E)(\mathbb{Q}, T) + \#(I \setminus E)(\mathbb{Q}, T) = o(T)$ , which is clearly false.

2  $\Rightarrow$  3. By Hieronymi and Miller [18, 1.4],  $E$  has Assouad dimension zero (see [18, Section 4] for the definition). Thus, given  $\epsilon > 0$ , there exists  $C > 0$  such that for all  $T > 1$

$$\#E(\mathbb{Q}, T) \leq \text{net}_{1/T}(E \cap [-T, T]^n) \leq C \left( \frac{T}{1/T} \right)^{\epsilon/2} = CT^\epsilon.$$

3  $\Rightarrow$  1. If  $E \subseteq \mathbb{R}$ , then  $E^{\text{trans}} \subseteq E \setminus \text{int}(E)$ ; if  $E$  is definable, then so is  $E \setminus \text{int}(E)$ .  $\square$

There are several classes of structures known to satisfy condition 4.2.2 that are not o-minimal. It would take us too far afield to discuss them here, but see [18, 23] for examples and references. Remarkably, the following questions seem to be open: Does Condition 4.2.2 imply that  $\#E^{\text{trans}}(\mathbb{Q}, T)$  is sub-polynomial for every definable set  $E$ ? If  $E \subseteq \mathbb{R}^n$  is a boolean combination of open sets and  $(\mathbb{R}, +, \cdot, E)$  does not define  $\mathbb{Z}$ , is  $\#E^{\text{trans}}(\mathbb{Q}, T)$  sub-polynomial?

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