Meshing real algebraic sets with singularities.

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Abstract

The aim of this article is to describe an algorithm to compute a mesh isotopic to the real zero set Z of a given polynomial with real coefficients in two or three dimensional space. Though there are already several methods able to deal with smooth manifolds, the actual treatment of singular ones is not so effective. The new algorithm that we propose handles the singularities using a combination of tools coming from singularity theory and real algebraic geometry. Using the notion of Whitney stratification and Milnor balls, we effectively manage to test if the zero set has a conic structure in a given box. With this criterion, we compute a finite partition of the space in boxes so that the set Z in each box has the same topology as a cone and then mesh the surface in these boxes. We finally obtain a triangulation of our set which is compatible with some Whitney stratification. Actually, most of the algorithm we present here works for semialgebraic sets. The limiting part, both computationally and theoretically, is the effective computation of a Whitney stratification for the set we want to mesh. For this matter, we rely on an existing algorithm computing such stratifications, for curves and surfaces defined as the zero set of a single polynomial. We detail the algorithm in this context and show examples at the end.

1 Introduction

In the following, n denotes the number of variables $X_1, ..., X_n$ and $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$. For any set of polynomials $F \subseteq \mathbb{K}[X_1, ..., X_n]$, let $V(F)$ be the zero set of F: $\{x \in \mathbb{K}^n : \forall f \in F, f(x) = 0\}$. For any set $E \subseteq \mathbb{K}^n$, we define $\dim(E)$ as the topological dimension of E, that is $\sup\{\dim(M) : M \subseteq E \text{ smooth manifold}\}$.

Remark 1.1 *For this definition of the dimension, we follow [**?**]. Note that when E is an algebraic set, the topological dimension and the Krull dimension coincide. If E is a semialgebraic set, the topological dimension may be defined for instance as the supremum of the dimensions of the cells (diffeomorphic to hypercubes) in a CAD (Cylindrical Algebraic Decomposition) of E.*

The problem that we will consider is the following:
Given $n \in \mathbb{N}^*$, $f \in \mathbb{R}[X_1, ..., X_n]$ and $C \subseteq \mathbb{R}^n$ a hypercube, can we compute effectively and efficiently a finite triangulation T such that T is isotopic to $V(f) \bigcap C$?
By isotopic we mean the following:

Definition 1.2 (Isotopy) *$\forall E, F \subseteq \mathbb{K}^n$, E is said to be isotopic to F iff $\exists h : [0,1] \times E \to \mathbb{K}^n$ such that h is continuous, $h(0, E) = E$, $h(1, E) = F$ and $\forall t \in [0,1]$, $h(t, .)$ is a homeomorphism on its image.*

It is well known that such a triangulation does exist (see for instance [?, ?, ?]). The most effective way to prove its existence is to deduce it from the existence of a CAD (Cylindrical Algebraic Decomposition) for semialgebraic sets (see [?, ?] for the existence of such a decomposition). As a matter of fact, once one has a CAD for a semialgebraic set, it is possible to find a triangulation of that set using the description of the cells and of their connections in the CAD (see [?]). Unfortunately there are two problems with this approach.

The first problem is that the computation of a CAD, though utterly effective, is far from efficient. From a theoretical viewpoint, the CAD algorithm is doubly exponential in n, the number of variables. Thus one cannot expect any good asymptotic behavior. From a practical viewpoint the algorithm computes lots of polynomials with a high degree even for $n = 3$. Thus even for polynomials with few variables, the CAD algorithm is very slow.

The second problem is that the way one can compute a triangulation from the CAD representation is not clearly effective. The proof of the existence of a triangulation using the CAD in [?] is not straightforwardly constructive. Moreover, this Cylindrical Algebraic Decomposition does not yield directly the topology, since the adjacency relations between the cells are not known. This non-trivial task is partially achieved in [?].

Another approach has been proposed recently in [?] for computing the topology of surfaces $Z = V(f)$ in $\mathbb{R}^3$. Exploiting the properties of Whitney stratifications, it requires to compute the topology of the polar curve of Z for one direction in $\mathbb{R}^3$ and the topology of sections of the surface at critical values, which are depending on algebraic numbers. The method that we describe in this paper applies for arbitrary dimension, once we have computed a Whitney stratification and avoid cascaded computation with algebraic numbers.

Triangulations of algebraic varieties are already used in practice (in computer aided geometric design for instance, [?]). Unfortunately the algorithms used in this context focus on the visual quality of the rendering and efficiency, and not on the topology certification. At best the mesh is guaranteed to be isotopic to the zero set of the polynomial f when $V(f)$ is smooth. At worst there is no guarantee at all for the topology of the computed mesh. This is the case for instance for the marching cube algorithm [?]. One can make it produce arbitrarily many triangles by setting the grid step smaller and smaller. But there is never any guarantee that the grid step is small enough. The same problem occurs in other algorithms such as [?], [?], [?], [?]. So, these algorithms do not suit our need for topological correctness.

Our algorithm is half way from these two very different approaches. It does produce a mesh isotopic to $V(f)$ in a rather efficient way. Thus we keep all the topological information. It produces a mesh that is arbitrarily close to $V(f)$ in terms of Hausdorff distance to $V(f)$. This accounts for the visual quality of the mesh. A characteristic of our method, is to be able to handle singular curves and surfaces, which is not the case for other subdivision methods such as [**?**]. More precisely, the subdivision criteria is based on a local conic structure test, which allows us to triangulate a surface, around a singular point. For this purpose, we compute a Whitney stratification of $V(f)$. From this stratification, we deduce where to compute the vertex of a potential local conic structure in an hypercube. If the local conic structure test is not valid, we subdivide the hypercube, and repeat this process until we get the topology of the hypersurface in each hypercube. By choosing adequately the position where we split, and computing a triangulation of the hypersurface compatible with the local conic structure, we show that this algorithm produces a triangulation of the hypersurface, which is isotopic to it, and compatible with our Whitney stratification, ie the strata are union of simplices.

Note that the theoretical question of finding a Whitney triangulation of a (say) subanalytic set $X \subset \mathbb{R}^n$, ie a triangulation such that each simplex is a stratum of a Whitney stratification of X was a difficult question open for a long time (at least for $dim(X) > 2$) and recently solved by M. Shiota (Preprint).

Milnor balls are usually consider for the Euclidean distance. In this paper, by considering hypercubes, instead of hyperspheres, and by proving Milnor ball properties for such hypercubes, we present a new type of methods. Being able to precompute characteristic points, where the strata of a Whitney stratification are critical for the projection along the axis, we simplify the test of local conic structure, which yields an efficient subdivision algorithm, even for singular varieties.

Currently, there is one main drawback to the algorithm. There is no efficient way to compute Whitney stratifications for any algebraic variety [**?**]. There only exists an algorithm for curves and surfaces in $\mathbb{R}^3$. It is described in [**?**] (see also [**?**]) and uses a method close to the one of the CAD algorithm. But it is efficient, since it makes minimal use of projective resultants, as opposed to the CAD algorithm that computes lots of resultants and subresultants.

This is the direction that is left to be investigated to make a complete, efficient, certified algorithm.

2 Using the local conic structure of semialgebraic varieties.

In this section, we consider the general framework of semialgebraic sets: we assume we are given a semialgebraic set Z defined by a finite set of polynomial equalities and inequalities and a hypercube C. The algorithm that we are going to describe partitions C into finitely many smaller hypercubes $(C_i)_i$ so that for all i, $Z \bigcap C_i$ is isotopic to a cone.

Here are more precise statements and definitions. $\forall E \subseteq \mathbb{R}^n$

- The interior of E is denoted by $\overset{\circ}{E}$. The closure of E is denoted by $\overline{E}$. The boundary of E is denoted by δE.
- $\forall E \subseteq F \subseteq \mathbb{R}^n$, $\forall (p_n)_{n\in\mathbb{N}} \subseteq F$, $\forall p \in F$,
  $p_n \overset{E}{\longrightarrow} p$ means that $(p_n)_{n\in\mathbb{N}}$ converges to p in E.
- $\forall p, q \in \mathbb{R}^n$, define $\overline{pq}$ as the line passing through p and q, and define $[pq]$ as the line segment joining p to q.
- $\forall a \in \mathbb{R}^n$, $\forall B \subseteq \mathbb{R}^n$, $a \star B$ denotes the cone with vertex a and base B, that is $\bigcup_{x\in B} \overline{px}$.
  Also define $[a \star B]$, the real positive cone with vertex a and base B, as $\bigcup_{x\in B} [px]$.

Definitions 2.1 (Hypercubes and cube partitions)

- *A closed hypercube C, is a set C of the form $C = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$ where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, $n \in \mathbb{N}$.*
  An open hypercube is the interior of a closed hypercube. If $n = 0$, then $\{0\}$ is both a closed and open hypercube.
- *A cube partition of a hypercube C is a partition Γ of $C = \bigcup_{\gamma\in\Gamma} \gamma$, where every cell $\gamma \in \Gamma$ is an open hypercube.*

For a real number $r > 0$, a point $p \in \mathbb{R}^n$ and a set $C \subset \mathbb{R}^n$, we denote by $p + \rho C$ the image of C by a homothety of center 0 and factor ρ, and then, a translation of vector p.

Definitions 2.2 (Conic structure and conic cube partition)

- *Let C be a closed hypercube. A set $E \subseteq C \subseteq \mathbb{R}^n$ is said to have conic structure in C iff E isotopic to $[p \star (E \bigcap \delta C)]$ where $p \in \overset{\circ}{C}$.*
  Clearly this definition does not depend on the point p that is chosen in $\overset{\circ}{C}$.
- *Let $C \subseteq \mathbb{R}^n$ be a hypercube, Γ a cube partition of C, and $E \subseteq C$. Then Γ is said to be a conic cube partition of C for E iff $\forall \gamma \in \Gamma$ cell, $E \bigcap \overline{\gamma}$ has conic structure in $\overline{\gamma}$.*

It is likely that conic cube partitions exist for semialgebraic sets thanks to the following theorem (see [**?**] [**?**]).

Theorem 2.3 (Local conic structure)

$\forall Z$ semialgebraic variety, $\forall p \in Z$, $\forall N : \mathbb{R}^n \to \mathbb{R}^+$ semialgebraic mapping such that $N^{-1}(0) = \{p\}$, $\exists \epsilon > 0$ and $\exists h : N^{-1}([0, \epsilon]) \bigcap Z \to [p \star (N^{-1}(\epsilon) \bigcap Z)]$ semialgebraic homeomorphism, such that $\forall x \in N^{-1}([0, \epsilon]) \bigcap Z$, $N(h(x)) = N(x)$ and $h|_{N^{-1}(\epsilon) \bigcap Z} = Id$.

To use this theorem in our case, it is necessary to define a suitable $N : \mathbb{R}^n \to \mathbb{R}^+$.

Definition 2.4 (Convex set gauge) *$\forall p \in \mathbb{R}^n$, $\forall C \subseteq \mathbb{R}^n$ convex set such that $p \in \overset{\circ}{C}$, define the convex set gauge $N_{p,C}$ of (p, C), so that : $\forall x \in \mathbb{R}^n$, $N_{p,C}(x) = inf\{\alpha > 0 : \frac{x-p}{\alpha} + p \in C\}$.*

Notice that $\mathrm{Graph}(N_{p,C}) = (0,0)\star(\delta C, 1) \bigcap \mathbb{R}^n \times \mathbb{R}^+$. In particular, $N_{p,C}^{-1}([0,1]) = C$ and that $N_{p,C}^{-1}(\{0\}) = \{p\}$.
So, this naturally defines a convex gauge for any hypercube with a point inside. This is indeed a suitable candidate for the function N of theorem **??** thanks to the following proposition, which is a consequence of Tarski-Seidenberg quantifier elimination theorem [**?**].

Proposition 2.5 (Semialgebraic convex set gauge is semialgebraic) *$\forall p \in \mathbb{R}^n$, $\forall C \subseteq \mathbb{R}^n$ semialgebraic convex set such that $p \in \overset{\circ}{C}$, $N_{p,C}$ is a semialgebraic mapping.*

Proof. $\forall p \in C \subset \mathbb{R}^n$, the graph of $N_{p,C}$ is defined by the following condition: $\forall (a, b) \in \mathbb{R}^n \times \mathbb{R}$, $(a, b) \in \mathrm{Graph}(N)$ <u>iff</u> $\forall \lambda > b$, $\frac{a-p}{\lambda} + p \in C$ and $\forall \lambda \in]0, b[$, $\frac{a-p}{\lambda} + p \notin C$.

As C is semialgebraic, being contained in C is a semialgebraic condition. Hence the condition is a semialgebraic first order formula and the graph of N is semialgebraic as the quantifiers can be eliminated. □

Clearly hypercubes are semialgebraic sets. Hence the local conic structure theorem applies. This guarantees that, as C is compact, for a semialgebraic set Z in a hypercube C, one can find a finite covering of $Z \bigcap C$ with hypercubes with different sizes and positions such that Z has a conic structure inside each hypercube.

But the structure of cube partitions is much stiffer: the hypercubes interiors can't overlap. As we are considering a relatively compact subset of Z (that is $Z \bigcap C$) we hope to get a cube partition, provided that the cells are small enough. It has been proved that this is the case for smooth varieties []. We haven't yet proved it in the general case, hence we can only resort to a heuristic to generate cube partitions that "should" work. In section **??** we give such a heuristic.

In the next section we say how we test that a given cube partition is a conic cube partition.

3 Stratification and conic structure

We introduce the notion of Whitney stratification of semialgebraic varieties and define Milnor balls that are compact convex sets meeting some transversality conditions. Then we prove that indeed, Z has conic structure in Milnor balls. In section **??**, we explain how the transversality conditions can effectively be tested when the "ball" is a box. This gives rise to an effective criterion for conic structure.

3.1 Stratification of the variety.

A stratification of Z is basically a partition of Z into a set of smooth submanifolds called strata. This notion first clearly emerged in [**?**].

There are many ways to choose stratifications. Consider the example of two lines crossing at the origin. One could choose the following stratification: the first stratum is the first line, and the second stratum is what is left of the second line. However, the stratifications such that, intuitively, the strata connect smoothly to each other have better properties. For instance, the previous stratification of the two lines is quite unnatural as the origin is by no way special in the first stratum.

Such a condition of "smooth connection" between strata has been made precise by Whitney and is known as Whitney (b) condition.
$\forall \sigma$ manifold, $\forall p \in \sigma$, define $T_p(\sigma)$ as the tangent space to σ at p.

Definition 3.1 (Whitney (b) condition) *$\forall E \subseteq \mathbb{K}^n$, $\forall \sigma_1, \sigma_2 \subseteq E$, $\forall p \in \sigma_1 \bigcap \sigma_2$, one says that (σ_1, σ_2) satisfies the Whitney (b) condition at p iff $\forall p_n \xrightarrow{\sigma_1} p$, $\forall q_n \xrightarrow{\sigma_2} p$, if $\lim_{n\to\infty} \overline{p_n q_n} = l$ and $\lim_{n\to\infty} T_{q_n}(\sigma_2) = \tau$ then $l \subseteq \tau$.*

Remark 3.2 *When taken over $\mathbb{C}$ the lines $\overline{p_n q_n}$ are complex lines. Though we are considering real algebraic varieties, it is useful to resort to the complex case to prove the correctness of the algorithm computing Whitney stratifications described in [**?**].*

We now make precise what is meant by partition of Z in smooth submanifolds meeting the Whitney (b) condition, and we come up with the definition of Whitney (b)-regular stratifications.

Definition 3.3 (S-decomposition) *$\forall E \subseteq \mathbb{K}^n$, $\forall \mathcal{S}$ partition of E, $\mathcal{S}$ is called an S-decomposition of E iff*

- *every stratum $\sigma \in \mathcal{S}$ is smooth,*
- *$\forall \sigma_1, \sigma_2 \in \mathcal{S}$, $(\sigma_1 \bigcap \overline{\sigma_2} \neq \emptyset) \Rightarrow \sigma_1 \subseteq \overline{\sigma_2}$*
- *$\forall K \subseteq \mathbb{R}^n$ compact, $\{\sigma \in \mathcal{S} : \sigma \bigcap K \neq \emptyset\}$ is finite.*

Remark 3.4 *We do not require the strata to be connected components. This is because, on the one hand, there is no need to consider connected components computationally (and it would be difficult to do so). On the other hand, in the proofs, we consider local problems, on one of the connected components.*

Definition 3.5 (Whitney (b)-regular stratification) *$\forall E \subseteq \mathbb{K}^n$, $\forall \mathcal{S}$ S-decomposition of E, $\mathcal{S}$ is a Whitney (b)-regular stratification (or Whitney stratification for short) of E iff it satisfies the two following conditions:*

- *$\forall \sigma \in \mathcal{S}$, σ smooth and locally closed.*
- *$\forall \sigma_1, \sigma_2 \in \mathcal{S}$ such that $\sigma_1 \subseteq \overline{\sigma_2}$, $\forall p \in \sigma_1$, (σ_1, σ_2) satisfies the Whitney (b) condition at p.*

Example 3.6 *Let C be a closed hypercube, then the cells of a cube partition of C form a Whitney (b)-regular stratification of C.*

The Whitney (b) condition has the nice property that it naturally stratifies the semialgebraic varieties (see [**?**]). That is, starting with the smooth and singular part of the variety, then recursively considering the set of points where the Whitney (b) condition is not satisfied yields indeed a Whitney stratification. Hence semialgebraic sets are Whitney (b)-regularly stratifiable.

3.2 Milnor balls and conic structure

In this section we show how stratifying Z will help us to decide whether Z has conic structure in a hypercube C. For that purpose, we exploit the concept of Milnor ball.

Definition 3.7 (Transverse manifolds) $\forall M, N \subseteq \mathbb{R}^n$ *smooth manifolds, M and N are said to be transverse iff $\forall p \in M \bigcap N$, $T_p(M) + T_p(N) = \mathbb{R}^n$.*

$\forall X, Y \subseteq \mathbb{R}^n$, X and Y are said to be transverse iff $\forall p \in X \bigcap Y$, $\exists M \subseteq X, N \subseteq Y$ manifolds containing p such that M and N are transverse in the previous sense.

Remark that if $M \bigcap N = \emptyset$ then M and N are transverse.

Remark 3.8 *Coming back to Whitney stratified varieties, if V_1, V_2 are two varieties Whitney (b)-regularly stratified by $\mathcal{S}_1$ and $\mathcal{S}_2$ and that $\forall(\sigma_1, \sigma_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, σ_1 and σ_2 are transverse, then $V_1 \bigcap V_2$ is Whitney stratified by $\mathcal{S} := \{\sigma_1 \bigcap \sigma_2 : (\sigma_1, \sigma_2) \in \mathcal{S}_1 \times \mathcal{S}_2\}$*

Definition 3.9 (Milnor ball) $\forall Z \subseteq \mathbb{R}^n$ *analytic variety such that Z admits a Whitney stratification $\mathcal{S}_Z$, $\forall C \subseteq \mathbb{R}^n$ compact convex set such that δC admits a Whitney stratification $\mathcal{S}_{\delta C}$, $\forall p \in \overset{\circ}{C}$, C is called a Milnor ball for Z with vertex p iff $\forall t \in]0, 1]$, Z is transverse to ${N_{p,C}}^{-1}(t)$.*

We are going to prove now that these Milnor balls have a conic structure, using Thom-Mather theorem (see [**?**]).

Definition 3.10 (Proper stratified submersion) $\forall n, m \in \mathbb{N}$, $\forall f : \mathbb{R}^n \to \mathbb{R}^m$, *for all semi-algebraic set $Z \subseteq \mathbb{R}^n$, for all Whitney stratification $\mathcal{S}$ of Z, we say that f is a proper stratified submersion for $\mathcal{S}$ iff $f|_Z$ proper and $\forall \sigma \in \mathcal{S}$, the differential map $\partial f|_\sigma$ has constant rank.*

Theorem 3.11 (Thom-Mather) $\forall Z \subseteq \mathbb{R}^n$, $\forall \mathcal{S}$ *Whitney stratification of Z, $\forall g : \mathbb{R}^n \to \mathbb{R}^m$ such that g smooth proper stratified submersion for $\mathcal{S}$,*
$\exists h : Z \to \mathbb{R}^m \times (g^{-1}(0) \bigcap Z)$ stratum preserving homeomorphism, such that h smooth on each stratum and $\Pi_{\mathbb{R}^m} \circ h = g$.

The following lemma, introduced in [**?**], is a direct consequence of theorem **??**. We will not use it explicitly in the next proof but we mention it because we use the same idea in the proof of corollary **??**.

Lemma 3.12 (Moving the Wall, [?] theorem 1, p.72) *$\forall Z \subseteq \mathbb{R}^n$ Whitney stratified by $\mathcal{S}_Z$, $\forall g : \mathbb{R}^n \to \mathbb{R}^m$ smooth map,*
$\forall Y \subseteq \mathbb{R}^m \times \mathbb{R}$ (closed) Whitney stratified by $\mathcal{S}_Y$, let $\Pi_\mathbb{R}$ the projection to the second component and $Y_t = \Pi_\mathbb{R}^{-1}(t) \bigcap Y$.
If $\Pi_\mathbb{R}$ is a stratified submersion for $\mathcal{S}_Y$ and proper on Y, and $\forall t \in \mathbb{R}$, $\mathrm{Graph}(g|_Z)$ and $\mathbb{R}^n \times Y_t$ are transverse,
then $\exists h : g^{-1}(Y_0) \bigcap Z \to g^{-1}(Y_1) \bigcap Z$ stratum preserving homeomorphism, such that h smooth on each stratum.

Definition 3.13 (Link) *$\forall Z \subseteq \mathbb{R}^n$, $\forall g : \mathbb{R}^n \to \mathbb{R}^+$, define $\mathrm{Link}_Z(g)$ as $\mathrm{Graph}(g|_Z) \subseteq Z \times \mathbb{R}$.*

Now we can prove that Milnor balls have conic structure, as a corollary of theorem **??**. This corollary immediately applies to semialgebraic sets.

Corollary 3.14 (Milnor balls have conic structure) *$\forall Z \subseteq \mathbb{R}^n$ which admits a Whitney stratification $\mathcal{S}$, $\forall C \subseteq \mathbb{R}^n$ Milnor ball for Z, Z has a conic structure in C.*

Proof. To prove the corollary, we consider $\mathrm{Link}_Z(N_{p,C}) = \{(x,t) \in Z \times \mathbb{R} : N_{p,C}(x) = t\}$. We are going to prove first the topological triviality of $\mathrm{Link}_Z(N_{p,C})$ over over the interval $]0, 1+\eta[$, for $\eta > 0$ sufficiently small. Then using Thom-Mather theorem, we explicitly build an isotopy from $C \bigcap Z$ to $[p \star \delta C]$ by continuously deforming $N_{p,C}^{-1}(t) \bigcap Z = \Pi_\mathbb{R}^{-1}(t) \bigcap \mathrm{Link}_Z(N_{p,C})$ to $\Pi_\mathbb{R}^{-1}(1) \bigcap \mathrm{Link}_Z(N_{p,C}) = \delta C$.

First of all, being transverse is an open condition. Therefore if C is a Milnor ball, as C is compact, this implies that there exists an $\eta > 0$ such that $N_{p,C}^{-1}(t)$ ($\forall t \in]0, 1+\eta[$) is also transverse to Z. We recall that $\mathrm{Graph}(N_{p,C}) = (0,0) \star (\delta C, 1) \bigcap \mathbb{R}^n \times \mathbb{R}^+$. In particular $N_{p,C}^{-1}(1) = \delta C$ and $N_{p,C}^{-1}([0,1]) = C$.

Let $V := \mathrm{Link}_Z N_{p,C} \bigcap (\mathbb{R}^n \times]0, 1+\eta[)$. Let $\mathcal{S}_C$ a stratification of δC and $\mathcal{S}_Z$ a stratification of Z. We prove that V is Whitney stratified.

$$\begin{array}{rcl} V & = & \mathrm{Graph}(N_{p,C}|_Z) \bigcap (\mathbb{R}^n \times]0, 1+\eta[) \\ & = & \mathrm{Graph}(N_{p,C}) \bigcap (Z \times \mathbb{R}) \bigcap (\mathbb{R}^n \times]0, 1+\eta[). \\ & = & \mathrm{Graph}(N_{p,C}) \bigcap (Z \times]0, 1+\eta[) \end{array}$$

$\mathrm{Graph}(N_{p,C})$ is Whitney stratifiable because the gauge of a semialgebraic convex set (that is C) is semialgebraic hence Whitney stratifiable. Actually $\mathrm{Graph}(N_{p,C}) \bigcap \mathbb{R}^n \times]0, 1+\eta[$ is stratified by $\{(p,0) \star (\sigma_C, 1) \bigcap \mathbb{R}^n \times]0, 1+\eta[\,:\, \sigma_C \in \mathcal{S}_C\}$.

The Milnor ball hypotheses imply that $\mathrm{Graph}(N_{p,C}) \bigcap]0, 1+\eta[$ is transverse to $Z \times]0, 1+\eta[$. By remark **??**, $V = \mathrm{Graph}(N_{p,C}) \bigcap (Z \times]0, 1+\eta[)$ is Whitney stratified by the intersections of the strata, because it is the transverse intersection of two Whitney stratified sets. We call $\mathcal{S}_V$ this stratification.

Now, we apply theorem **??** to prove the topological triviality of V. Let $\Pi_\mathbb{R}$ the projection onto the image of $N_{p,C}$. $\Pi_\mathbb{R}$ is a projection hence it is smooth and it is proper on V as, for K compact, $N_{p,C}^{-1}(K)$ is bounded and closed. As we are projecting V onto $\mathbb{R}$, we only have to find, $\forall s = (x,y) \in V$, a direction

$d \in T_s(V)$ (that is $d \in T_s\sigma$, $s \in \sigma \in \mathcal{S}_V$) such that $\partial\Pi_{\mathbb{R}}(d) \neq 0$ (where $\partial\Pi_{\mathbb{R}}$ is the differential map of Π).

As $s \in V$, there exists $\sigma_S \in \mathcal{S}_Z$ and $\sigma_C \in \mathcal{S}_C$ such that $s \in \sigma_Z \times]0, 1+\eta[$ and $s \in (p,0) \star (\sigma_C, 1) \subseteq \text{Graph}(N_{p,C})$. As σ_C and σ_Z are transverse, $\dim T_x(\sigma_C) + \dim T_x(\sigma_Z) \geq n$. As the tangent spaces to Z and δC are extended in a new direction (that is $(0,\mathbb{R})$ for Z and $\mathbb{R}(p-s)$ for C), $\dim T_s((p,0) \star (\sigma_C, 1)) + \dim T_z(\sigma_Z \times]0, 1+\eta[) \geq n+2$. Because the ambient space has dimension $n+1$, there exists a direction d contained in both tangent spaces at s. Moreover, as the directions in which the tangent spaces are extended do not project to $\{0\}$ through $\partial\Pi_{\mathbb{R}}(=\Pi_{\mathbb{R}})$, we have $\partial\Pi_{\mathbb{R}}(d) \neq 0$. This means that $\Pi_{\mathbb{R}}$ has rank one at every $s \in V$. Hence $\Pi_{\mathbb{R}}$ is a smooth proper stratified submersion, and theorem **??** applies.

So, as $\delta C = N_{p,C}^{-1}(1)$, we proved that there exists an homeomorphism h mapping $(\delta C \bigcap Z) \times]0, 1+\eta[$ to V such that $\Pi_{\mathbb{R}} = \Pi_{\mathbb{R}} \circ h$ and $h|_{\{1\}\times(\delta C \bigcap Z)} = Id$.

We now prove that we have an isotopy between $[p \star \delta C]$ and $C \bigcap Z$.
Consider the following mapping.

$$\begin{array}{llll} \Phi_u : &]0,1] \times Z \bigcap \delta C & \to & \delta C \\ & t, q & \mapsto & H_{1/(t+(1-t)u)} \circ \Pi_{\mathbb{R}^n} \circ h(t+(1-t)u, \Pi_{\mathbb{R}} \circ h^{-1}(t,q)) \end{array}$$

Consider H_λ the homothety with center p and ratio λ. Then the following mapping is an isotopy.

$$\begin{array}{lllll} \Psi : & [0,1] \times Z \bigcap C & \to & C & \\ & u, q & \mapsto & H_{N_{p,C}(q)} \circ \Phi_u(N_{p,C}(q), q) & \text{if } q \neq p \\ & u, p & \mapsto & p & \text{otherwise} \end{array}$$

Indeed, it is clearly injective for all $u \in [0,1]$ and continuous. $\Psi(0, Z \bigcap C) = Z \bigcap C$ and $\Psi(1, Z \bigcap C) = [p \star \delta C]$. Moreover $\forall u \in [0,1]$, $N_{p,C}(\Psi(u,p)) = \mathbb{N}_{p,C}(\Psi(0,p))$ (ie. the isotopy preserves the strata). □

According to the previous lemma, to prove that a variety Z in a given hypercube has a conic structure it suffices to find a point p and prove that Z is a Milnor ball with vertex p. But to turn this remark into an algorithm, we have to specify how we choose p, so that the hypercube C is a Milnor ball for Z with vertex p.

One can remark that C will never be a Milnor ball with vertex p, if p is outside the lowest dimensional stratum of Z (among the strata entering C). Nevertheless, except for boxes containing strata of dimension 0, infinitely many possibilities remain when choosing p. However the following theorem hints that one should choose p as far from the border of the cube, as possible.

Proposition 3.15 (Milnor tube) $\forall Z \subseteq \mathbb{R}^n$ *Whitney stratified by* $\mathcal{S}_Z$, $\forall B$ *compact convex set such that* δB *is* <u>*smooth*</u> *and* $0 \in \overset{\circ}{B}$, $\forall \sigma \in \mathcal{S}_Z$, $\forall K \subseteq \sigma$ *compact set,* $\exists \Theta$ *such that,* $\forall \rho \in]0, \Theta[$, $\forall p \in K$, $p + \rho B$ *is a Milnor ball with vertex* p.

Proof. Notice first that as B is smooth, B is stratified by $\mathcal{S}_B = (\delta B, \overset{\circ}{B})$ and similarly for $p + \lambda B$, with $\lambda > 0$, $p \in \mathbb{R}^n$.

Let $M : p \in Z \mapsto \sup(\rho \in \mathbb{R}^+ : \forall \eta \in]0, \rho[,\ (p + \eta B)$ Milnor ball with vertex $p) \in \mathbb{R}$.

Suppose that there is a sequence $(p_i)_{i\in\mathbb{N}}$ such that $M(p_i) \longrightarrow 0$. Then there exists a sequence of balls $B_i = p_i + \lambda_i B$ such that $\forall i$, B_i is not a Milnor ball and $\lambda_i \longrightarrow 0$.

As the $(B_i)_i$ are not Milnor balls, the transversality condition is contradicted infinitely many times for pairs of strata $(\delta B_i, \sigma_i)_i$ at the points q_i, where $\sigma_i \in \mathcal{S}_Z$ a stratum of Z. As $\mathcal{S}_Z$ is a finite set, one can find an extraction ϕ such that $\forall i \in \mathbb{N}$, the pair of strata $(\delta B_{\phi(i)}, \beta)$ is not transverse at the point $q_{\phi(i)}$, where β is a fixed stratum of $\mathcal{S}_Z$. $\forall i$, $(q_{\phi(i)} - p_{\phi(i)})/\lambda_{\phi(i)} \in \delta B$ because by definition $q_{\phi(i)} \in \delta B_{\phi(i)}$. Because $p_{\phi(i)} \in K$ which is compact, and $(q_{\phi(i)} - p_{\phi(i)})/\lambda_{\phi(i)} \in \delta B$ that is compact too, one can find a subextraction ψ of ϕ such that $p_{\psi(i)}$ converges to a point $x \in K$, and that $(q_{\psi(i)} - p_{\psi(i)})/\lambda_{\psi(i)}$ converges to a vector $y \in \delta B$. Because $\lambda_{\psi(i)}$ converges to 0, $q_{\psi(i)}$ converges to the same point as $p_{\psi(i)}$, that is x.

The sequence of lines $\overline{p_{\psi(i)} q_{\psi(i)}}$ converges to $\mathbb{R}\, y$. Because the set of tangent spaces is compact, we can also assume that $T_i := T_{q_{\psi(i)}}(\beta)$ converges to τ. As δB is smooth (by hypothesis) $\dim T_{q_{\psi(i)}}(\delta B_{\psi(i)}) = n - 1$. As the transversality condition is not met at $q_{\psi(i)}$, we have $T_i \subseteq T_{q_{\psi(i)}}(\delta B_{\psi(i)})$ (otherwise the dimension of the sum would be strictly greater than $n - 1$ which contradicts the non transversality assumption). Therefore to the limit, we get $\tau \subseteq T_y(\delta B)$.

By hypothesis Z is Whitney (b) regularly stratified, hence the Whitney (b) condition is satisfied at x for (σ, β), thus $\mathbb{R}\, y \subseteq \tau (\subseteq T_y(\delta B))$. Because $0 \in \overset{\circ}{B}$, $y \in \delta B$ and B convex, $\mathbb{R}\, y \not\subseteq T_y(\delta B)$ which is a contradiction.

Therefore the sequence $M(p_i)$ cannot converge to 0 and is bounded from below on K by a constant Θ. $\square$

Intuitively this states that, when the ball B is smooth and sufficiently small, if one translates it along the stratum σ then the ball keeps being a Milnor ball, provided one stays at some distance from the border of the stratum (that is, the strata with lower dimension).

4 The algorithm

In this section, we detail the algorithm, based on the constructions of the previous section. We first describe how we generate a cube partition. Then we explain how to test that a hypercube has conic structure. In order to do this, we address the question of testing the transversality of two strata and describe how to choose a point on the lowest dimensional stratum entering the hypercube. We only shortly describe the algorithm computing a Whitney stratification for Z in section **??** and refer to [**?**] for more details. Finally, we describe an algorithm generating a mesh from a conic cube partition of Z.

4.1 Generating cube partitions

The following algorithm generates a cube partition of a hypercube C. The algorithm is adaptive, because it takes into account the complexity of the semi-algebraic set Z that we want to mesh. That is, the number of cells in the cube partition depends on the behavior of Z. The tamer Z, the fewer the cells.

Algorithm 4.1 (Generation of a cube partition)
INPUT: *a hypercube C and a semi-algebraic set Z described by a set of polynomial equations and inequalities.*

- *Let $I := \{C\}$ the initial cube partition, and $F = \emptyset$ the final cube partition.*
- *While $I \neq \emptyset$*
  - *Choose $D \in I$, remove D from I.*
  - *Test whether Z has conic structure in D (see section **??**). If so, add D to F. Otherwise, split D into several hypercubes $D_1, \ldots, D_k$ (see section **??**) and add them to I.*

The unspecified steps of this algorithm are described now.

4.2 Testing conic structure

In this section, we detail how we check the conic structure of Z in a hypercube C.

Algorithm 4.2 (Testing the conic structure)
INPUT: *a cube C, a semi-algebraic set Z described by a set of equations and inequalities, and a Whitney stratification $\mathcal{S}_Z$ of Z. Let $\mathcal{S}_C$ be a Whitney stratification of δC.*

- *Compute a Whitney stratification $\mathcal{S}_Z$ of Z.*
- *Among the strata $\in \mathcal{S}_Z$ that intersect C, find one that has minimal dimension, say σ.*
- *Find a point $p \in \sigma$.*
- *To test whether C is a Milnor ball with vertex p for Z, for every stratum $\sigma_C \in \mathcal{S}_C$ and $\sigma_Z \in \mathcal{S}_Z$, do*
  - *If $[p \star \sigma_C]$ is not transverse to σ_Z, return false (that is, "C is not a Milnor ball").*
  - *If it is transverse proceed to the next pair of strata to test.*
- *Return true (that is, "every pair of strata is transverse, hence C is a Milnor ball").*

There are three points in this algorithm, which should be detailed now:

- How do we compute a Whitney stratification for Z ?
- How do we find p on the lowest dimensional stratum ?
- How do we test that $[p \star \sigma_C]$ and σ_Z are transverse ?

4.2.1 Computing a Whitney stratification

In order to simplify the local conic structure test, we compute a Whitney stratification of Z, such that $\sigma^0 \cup \cdots \cup \sigma^{l-1}$ contains the critical points of $\overline{\sigma^l}$ for all the projections along the hyperplanes orthogonal to the axes. For an arbitrary dimension, it consists in refining the stratification described in [**?**] by considering simultaneously the projections in every direction.

For curves and surfaces in $\mathbb{R}^3$, a more effective way is to refine the method described in [**?**], by considering the polar variety in each direction x, y, z.

4.2.2 Selecting a point on the lowest dimensional stratum

We assume we are given a semialgebraic set Z, a hypercube C, and a Whitney stratification $\mathcal{S}_Z = \sigma^0, \ldots, \sigma^k$ of Z, where σ^i is the semi-algebraic strata (not necessarily connected) of dimension i. In order to compute a point on the lowest dimensional stratum, we enumerate the strata (σ^i) in increasing dimension order and test if there is a point in $\sigma^i \bigcap \overset{\circ}{C}$. In the general framework of semialgebraic sets, testing if $\sigma^i \bigcap \overset{\circ}{C}$ is empty, reduces to solving a zero-dimensional problem, such as, for instance, computing the closest point on a real algebraic hypersurface deduced from the equalities defining σ^i and the inequalities defining C (see [**?**, **?**]).

This can be made more effective if $Z = V(f)$ and $\mathcal{S}_Z$ is a stratification described in section **??**. Suppose that $\sigma^i \bigcap C = \emptyset$ for all $i < l$. Then, either σ^l enters and leaves C, that is, it cuts δC (at two different points at least). Or it never cuts δC so that if $\sigma^l \bigcap C \neq \emptyset$, the set of critical points for any direction has at least a point in C. But by construction of the stratification $\sigma^0 \cup \cdots \cup \sigma^{l-1}$ contains the critical points of σ^l for the projection in one of the axes directions. This is a contradiction since $\sigma^i \bigcap C = \emptyset$ for all $i < l$. In other words, if $\sigma^i \bigcap C = \emptyset$ for $i < l$ and $\sigma^l \bigcap \delta C = \emptyset$ then $\sigma^l \bigcap C = \emptyset$.

Thus, we can proceed by induction on the dimension as follows: if $l = 1$, we test is $\sigma^1 \cap \delta C$ is of dimension 0 or empty, by solving 0-dimensional problem in one dimension less. If $l > 1$ and for each face F of δC, test the intersection of σ^l with its boundary δF.

This gives rise to the following algorithm, for computing a point on a stratum, in the interior of an hypercube.

Algorithm 4.3 (Computing a point on a stratum)
INPUT: *a stratum σ of $\mathcal{S}_Z$ of dimension l and an hypercube C.*

- *If σ is of dimension* 0*, compute the points of $\sigma \cap C$ and check if there is one point in $\overset{\circ}{C}$.*

- *If σ is of dimension > 0, test if σ cuts δC. Re-apply this algorithm with $\sigma \cap \delta C$ which is of dimension $l-1$ and for C one of the faces of δC. (This recursion obviously ends as the calls are done on the faces, hence the dimension drops by one each time).*

- *If we do find a point q on $\delta C \bigcap \sigma$. apply a Newton-type method, following the tangent space of σ at q, in order to get a point $q' \in \sigma \cap \overset{\circ}{C}$.*

Remark that this algorithm may fail if the stratum σ is tangent to δC. This is another part of the algorithm where the choice of a cube partition whose cubes are transversal to the strata has great importance.

4.2.3 Testing if two strata are transverse

We want to test whether $[p \star \sigma_C]$ and σ_Z are transverse. To do this we take advantage of the particularly simple form of the following stratification of C : if $C = [a_1, b_1] \times \ldots \times [a_n, b_n]$, we stratify C in 3^n strata corresponding , for each direction i ($i \in \{1, \ldots, n\}$), to whether $x_i = a_i$ or $x_i \in]a_i, b_i[$ or $x_i = b_i$. Let $\mathcal{S}_C$ be this stratification of C. Then $\forall \sigma_C \in \mathcal{S}_C$, $p \star \sigma_C$ spans an affine hyperspace that we denote by $A(\sigma_C)$.

The algorithm to test whether C is a Milnor ball with vertex p is as follows.

Algorithm 4.4 (Transversality test)
INPUT: *a point $p \in \mathbb{R}^n$, a stratum σ_C of C, a stratum σ_Z of $\mathcal{S}_Z$.*

- *If $\dim(p \star \sigma_C) + \dim(\sigma_Z) < n$ then return true (that is the two strata are transverse). This is because, because of the dimension, the two sets will not meet generically. Else go to the next step.*

- *By a linear change of variables restrict the inequalities defining σ_Z to the affine hyperspace $A(p \star \sigma_C)$ (the smallest affine space containing $p \star \sigma_C$).*

- *Compute the set of critical points $\chi_{\Pi,C}$ of the projection in $A(p \star \sigma_C)$ of σ_Z along the directions of $A(\sigma_C)$ (this is meaningful as $A(\sigma_C) \subseteq A(p \star \sigma_C)$). $\chi_{\Pi,C}$ is a finite set because generically the dimension of the intersection $A(p \star \sigma_C) \bigcap \sigma_Z$ is $\sup(0, \dim(p \star \sigma_C) + \dim(\sigma_Z) - n)$. Hence $\dim \chi_{\Pi,C} = 0$ generically.*

- *Test whether $\chi_{\Pi,C} \bigcap [p \overset{\circ}{\star} \sigma_C]$ is empty by enumerating over the points in $\chi_{\Pi,C}$.*

  - *If not, the stratum σ_Z and the image of σ_C by a homothety with center p are not transverse, thus C is not a Milnor ball.*

  - *If there is no critical point in $[p \overset{\circ}{\star} \sigma_C]$, the strata σ_Z and σ_C are transverse.*

The correctness is obvious as this exactly tests the transversality condition in the definition of Milnor ball.

Remark 4.5 *It is possible to test directly that* $C \bigcap \overset{\circ}{p \star \sigma_C} \bigcap \chi_C$ *is empty by using Hermite's method. Hermite's method allows us to count the number of points in a half space, as* $[p \star \sigma_C]$ *is an intersection of half spaces, one can test whether the number of points in* $[p \overset{\circ}{\star} \sigma_C] \bigcap \chi_C$ *is 0 [?, ?].*

4.3 Splitting the hypercube

In order to guarantee that the algorithm stops, we have to check that the test of the local conic structure will eventually be positive. Since in this test, we are computing a point on the stratum of lowest dimension in the box and check transversal intersection from this point with the surface, we should avoid that one of the strata of the box is tangent to one of the strata of the hypersurface.

For this purpose, we precompute the points of the hypersurface, where the tangent space to the strata of codimension k are orthogonal to less that k axis directions. This yields a finite number of points in $\mathbb{R}^n$, which coordinates have to be avoided during the splitting.

Let us detail the construction for implicit plane curves and implicit surfaces in $\mathbb{R}^3$.

For a planar curve defined by an equation $f(x, y) = 0$, we compute the set A of points where $f(x, y) = 0$ and $\partial_x f(x, y) = 0$ or $\partial_y f(x, y) = 0$. These are the points which are x or y critical. Then we split the rectangles so that the edges of the rectangles that we create do not contain any point of A and that the curve does not pass through their corners. Thus we have ensured that the curve never comes tangent to the faces or the corners of the rectangle.

For a surface in $\mathbb{R}^3$ defined by an equation $f(x, y, z) = 0$, this is not as easy as for plane curves. First, as in the two dimensional case, we test that the corners of the cubes that we create do not contain a point of Z.

For edges and faces of the cube, we have to avoid that the smooth part and the singular locus are tangent to axes-direction. Instead of handling the problem directly in three-space, for each direction x_i parallel to the axes of the frame, we project the associated polar variety along x_i onto the coordinate planes. More precisely, we compute $\mathcal{P}_{i,j} := V(\text{res}_{x_j}(f, d_i f)$ where $\text{res}_{x_j}(f, \partial_i f)$ is the projective resultant in the variable x_j of f and $\partial_i f$. Remark that $\mathcal{P}_{i,j}$ contains the projection of the singular locus. Then we apply the computation of critical points described before for planar curves. This yields a set of coordinates to avoid when splitting the cube.

As we have projected the polar variety of the smooth strata, we guarantee that strata of dimension 2 are never tangent to the faces nor the edges of the cube. As we also have projected the singular locus, we also guarantee that the strata of dimension 1 are never tangent to the faces nor the edges of the cube. Finally, as the coordinates of the points of the strata of dimension 0 are among those that we have computed, we also guarantee that the strata of dimension 0 are not on the faces, edges or vertices of the cube.

Remark that these tests do not cost that much as the directions in which we project do not change from one cube to another. Hence we can compute the projections once for all.

4.4 Generating the mesh

The last point to address is how, given a conic cube partition Γ of a hypercube $C \subseteq \mathbb{R}^n$ for a semi-algebraic set Z, one can get a triangulation isotopic to Z. The following algorithm is quite straightforward and works provided that we can test whether a particular point is in Z or not. This is the case for semialgebraic varieties as it suffices to test whether the inequalities defining the variety are satisfied at the point.

Algorithm 4.6 (From a conic cube partition to a triangulation)
INPUT: *a hypercube C, a conic cube partition Γ of C.*
For each $\gamma \in \Gamma$ with maximal dimension in Γ,

- *If* $\dim \gamma = 0$ *(ie. $\gamma = \{p\}$ for some $p \in \mathbb{R}^n$)*
  - *If $p \in Z$, then $T_\gamma := \{\gamma\}$ is a triangulation of γ*
  - *else $T_\gamma := \emptyset$ is a triangulation of γ.*
- *If* $\dim \gamma > 0$
  - *Call recursively the algorithm to triangulate each cell included in the boundary of γ with the conic cube partition $\{\alpha \in \Gamma : \alpha \subseteq \delta\gamma\}$.*
    This yields a triangulation T' of $\delta\gamma \bigcap Z$.
  - *Then take any point $p \in \overset{\circ}{\gamma}$ (actually we can take $p \in Z \bigcap \overset{\circ}{\gamma}$ as we have computed a point on the lowest dimensional stratum) and for each simplex $s \in T'$ add $[p \star s]$ to T_γ. Using the hypothesis on the conic structure of γ we conclude that T_γ is isotopic to $Z \bigcap \gamma$.*

OUTPUT: $T := \bigcup \{T_\gamma : \gamma \in \Gamma$ *with maximal dimension in* $\Gamma\}$.

As the isotopy result holds for each $\gamma \in \Gamma$ with maximal dimension and that $\bigcup\{\overline{\gamma} : \gamma \in \Gamma\} = C$, we get the isotopy result for the whole hypercube. This ends the description of the algorithm.

5 Examples in dimension 2 and 3

In this section, we limit ourselves to hypersurfaces $Z = V(f)$ in a two or three dimensional space. We give some details on how a Whitney stratification can be computed in this case using a theorem by Speder following the approach in [**?**].

5.1 Implicit curves

We consider a planar curve Z, defined by an equation $f(x, y) = 0$.

In a two dimensional space, it is very easy to compute a Whitney stratification : A miminal one consists of the singular locus σ^0 and the smooth stratum $\sigma^1 = Z - \sigma^0$. As σ^0 is 0-dimensional, the Whitney condition is automatically satisfied.

Since we also need to check the local conic structure, we compute the set A of points satisfying $f(x, y) = 0$ and ($\partial_x f(x, y) = 0$ or $\partial_y f(x, y) = 0$) and assume that during the splitting steps, the edges and vertices of sub-rectangles avoid these points.

In order to compute a point on the strata of lowest dimension in a rectangle C, we check whether C contains a point of A:

- If it contains no point of A, we compute one point on the curve in the interior of C.
- If it contains one point of A, we choose this point as a possible vertex of the local conic structure.
- If it contains two or more points of A, we split the rectangle, so that the maximal number of points of A in the subrectangles decreases strictly.

In order to test the local conic structure, we check whether the segments joining the chosen point and the vertices of the rectangles intersect the curve $Z = V(f)$.

5.2 Implicit surfaces

We consider here a surface Z of $\mathbb{R}^3$ defined by the equation $f(x, y, z) = 0$.

The algorithm for computing a Whitney stratification of Z uses the polar variety associated to a projection π on a plane, that we denote by χ_π. By a result of Speder [**?**], we know that for a generic projection the polar variety $(\overline{\sigma^1} = \chi_\pi)$ and the smooth stratum $(\sigma^2 = Z - \overline{\sigma^1})$ satisfy the Whitney B condition. Then we define σ^0 as the fibers above the singular points of $r(x, y) = \mathrm{res}_z(f, \partial_z(f))$ (where res_z is the resultant with respect to the variable z) defining $\pi(\chi_\pi)$. For π generic, the singular points in χ_π are contained in σ^0, hence $\sigma^1 = \chi_\pi - \sigma^0$ is smooth.

The Whitney condition between the strata (σ^2, σ^0) and (σ^1, σ^0) is automatically satisfied because σ^0 only contains isolated points. And the pair (σ^2, σ^1) satisfies the Whitney condition because of Speder's theorem. Therefore we have indeed computed a Whitney stratification for Z. Remark that if $Z = V(f)$ the objects we have defined are always algebraic with a low complexity, that is, only a few polynomials are necessary to define the σ^i.

In order to split the cubes and to test the local conic structure, we compute the projections of the polar varieties in the direction x, y, z onto the coordinate planes. This is performed by computing the resultant of f and its derivatives with respect to one of the variables. Taking, the square free part r of the resultant, we compute the points where this bivariate polynomial and one of its derivative vanish. This yields the two dimensional points that we lift onto the corresponding polar variety. For that purpose, we exploit the Sylvester matrix properties involved in the resultant computation (see [**?**] for more details). This set of points of $\mathbb{R}^3$ is denoted by A.

To choose a point on the strata of lowest dimension in a cube C,

- first we check if the cube C contains a point of A and if yes,we chose this point.
- otherwise, we check if one of the polar variety of Z in one direction intersects the cube, by intersecting it with the faces of the cube. If yes, we choose one point on this polar variety in the interior of the cube.
- if not, we check that the surface intersects at least two of the edges of C and compute one point on it in the interior of C.
- if none of these conditions are satisfied, then $C \cap Z$ is empty.

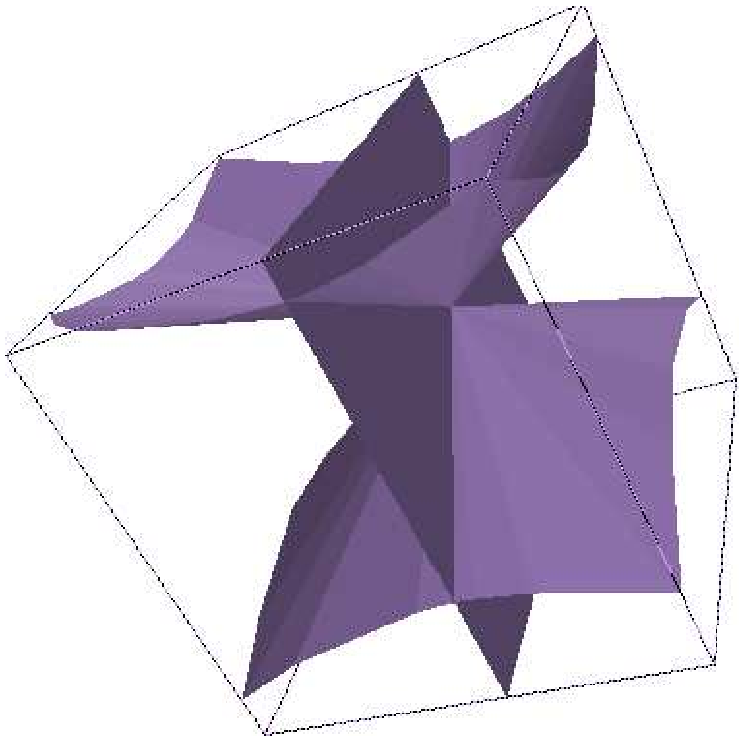
This yields a point p on the stratum of lowest dimension, if $C \cap Z$ is not empty.

Finally, to test the local structure, we check whether the cube C contains at most one point of A and whether the segments joining the chosen point p and the vertices of the cube intersect the surface Z inside C. Since a cube contains at most one point of A, the strata of the cube of dimension ≥ 1 will be transverse to the strata of Z and we only have to test that the segments joining p to the vertices of C do not intersect Z.

To illustrate the approach, we present hereafter two examples. A first example of a cone intersecting a plane and a second example of three planes crossing. The first example has been set up to illustrate how a badly chosen cube partition can cause the algorithm to fail. The second example works fine.

5.2.1 A first example

In this section we consider $f := x(x^2 + y^2 - 4z^2)$ and the initial cube $C := [-1, 1]^3$. $Z := V(f)$ is a cone cut by a plane.



The algorithm begins with the computation of the representation of a Whitney stratification as described in section **??** and [**?**]. We detail the steps of the algorithm on this example. The algorithm actually computes a Whitney stratification for the complexification of the variety, and what is proved in [**?**] is that the restriction to $\mathbb{R}^n$ of this stratification is still a Whitney stratification of the restriction of $V(f)$ to $\mathbb{R}^n$.

- The closure of the first stratum in $\mathbb{C}$ is the manifold itself as we are working on a complex hypersurface. So $\overline{\sigma^2} = V(f)$ where $f = x(x^2 + y^2 - 4z^2)$.
- We assume that the projection along the x direction is generic (we can assume this after a generic linear change of coordinates). We also assume that f is a squarefree polynomial (which we can assume as computing the squarefree part of a polynomial is an easy computation).
  Therefore the union of the polar variety and the singular locus can be easily described as the points where f and $\partial_x f$ vanish. We over approximate this set by the preimage of its projection along the x axis. We compute this projection using a projective resultant. We compute the squarefree part of this resultant both to get rid of unecessarily high degrees and because we are going to once again consider the singular points of this projection. So we get $\overline{\sigma^1} = V(f, f_1)$ with $f_1 = \text{Square free}\,\text{Res}_x(f, \partial_x f) = (y-2z)(y+2z)$.

- Finally we want to remove the singular points of the polar variety union the singular locus. We over approximate this set by the fibers over the singular points in the projection along the y axis. We compute it using a projective resultant. So we get $\sigma^0 = \overline{\sigma^0} = V(f, f_1, f_2)$ with $f_2 = \text{Square free}\,\text{Res}_y(f_1, \partial_y f_1) = z$.

One can see from this description that this is a CAD like method.

Here is the list of steps through which the algorithm goes after having computed the stratification.

- The algorithm tries a first cube partition : C itself, its faces, edges and corners.
- It starts with testing whether C is a Milnor ball.
  - It tries to find a point on the stratum intersecting C that has the lowest dimension.
  - It tries σ^0, and find the point $p := (0, 0, 0)$ inside C (actually $\sigma^0 = \{p\}$).
  - It tests the transversality for every pair of strata in $\mathcal{S}_C, \mathcal{S}_Z$
    - ∗ It chooses the stratum $\sigma_C := [-1, 1] \times [-1, 1] \times \{1\}$ in $\mathcal{S}_C$. The affine space containing $p \star \sigma_C$ is $\mathbb{R}^3$ itself.
      - · It chooses $\sigma^0 \in \mathcal{S}_Z$, that is $\{(0, 0, 0)\}$. It doesn't intersect $[p \overset{\circ}{\star} \sigma_C]$ hence it is transverse.
      - · It chooses σ^1 then σ^2, in both cases, there is no critical point for the projection along the directions of σ_C.
    - ∗ This is similar for any other stratum $\sigma_C \in \mathcal{S}_C$. $p = (0, 0, 0)$ is the only critical point for the projection of σ^0, and it obviously avoids $[p \overset{\circ}{\star} \sigma_C]$. There is no problem with σ^1 and σ^2 because they do not contain any critical points for the projections along the directions of σ_C.

  Finally, C is a Milnor ball with vertex p.
- Then the algorithm tests whether the faces have conic structure for $V(f)$.
  - We do not detail what the algorithm does on the faces where $V(f)$ is smooth, it handles them correctly.
  - The face $D := [-1, 1] \times \{1\} \times [-1, 1]$ is more interesting as $V(f) \bigcap D$ is a variety with a singularity. The algorithm tests if the variety defined by $g := x(x^2 - 4z^2 + 1)$ (which is the restriction of f to $A(D)$) is a Milnor ball. Assuming $V(f)$ and D are transverse (this is generically the case) one gets a whole stratification by restricting f_1 and f_2 to $A(D)$.
    - ∗ The algorithm finds a point on the stratum intersecting D that has the lowest dimension.

* It tries $\sigma^1 \bigcap D$ and finds $q := (0, 1, -\frac{1}{2})$.
* To test whether D is a Milnor ball with vertex q, it tests whether every pair of strata is transverse.
    · It chooses the stratum $\sigma_D := [-1, 1] \times \{1\} \times \{1\}$. It chooses the 0-dimensional stratum $\sigma^1 \bigcap D$ and the critical point of the projection $q' := (0, 1, \frac{1}{2})$ is in $[q \star \sigma_D]$. The transversality test fails for σ_D.

So D is not a Milnor ball.

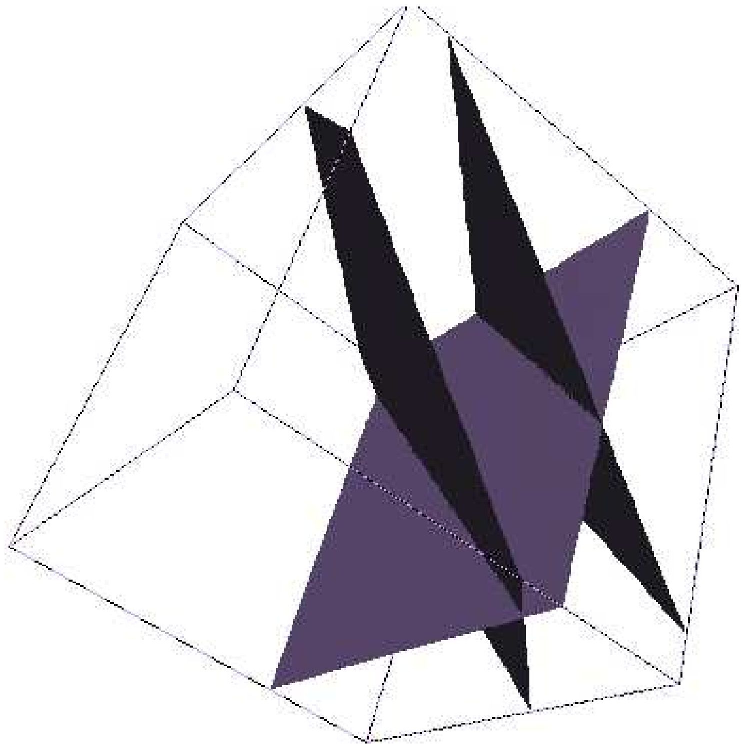
- Thus the algorithm tries to refine the cube partition by splitting D.

Now, if we don't split carefully the cube, problems may appear. For instance, in this case, if we split the face into 4 identical squares, the edges of the squares will coincide with Z, hence no matter how many times the square is split, the squares sharing the edge with $V(f)$ will never be Milnor balls. So the algorithm does not terminate. Using the computation described in section **??**, one can avoid such a problem.

Remark also that it would have been sufficient to translate the cube by $(\frac{1}{3}, 0, 0)$ for instance for the algorithm to succeed. If it had been done, the algorithm would have split the square until q and q' are separated and the Milnor ball test would have succeeded. This is how the picture above was obtained.

5.2.2 A second example

Now we consider $f := x(x + 5 + y + z)(x + y + z)$ and the initial cube $C := [-7, 5] \times [-7, 5] \times [-7, 5]$. This is a set of three planes.



The algorithm begins with the computation of the representation of a Whitney stratification. The steps of this computation are the same as in the first example.

- The closure of the first stratum in $\mathbb{C}$ is the manifold itself as we are working on a complex hypersurface. So $\overline{\sigma^2} = V(f)$ where $f = x(x+5+y+z)(x+y+z)$.
- We assume that the projection is generic and that f is squarefree.
The union of the singular locus and of the polar variety is described by $f = \partial_x f = 0$ and is over approximated by the preimage of its projection. So we get $\overline{\sigma^1} = V(f, f_1)$ with $f_1 = \text{Square free Res}_x(f, \partial_x f) = (y+z+5)(y+z)$.
- Finally we want to remove the singular points of the polar variety union the singular locus. We over approximate this set by the fibers over the singular points in the projection along the y axis that we compute using another resultant. And we get $\sigma^0 = \emptyset$ because $f_2 = \text{Square Free Res}_y(f_1, \partial_y f_1) = 1$.

So we have a Whitney stratification of the variety.

Here is the list of steps through which the algorithm goes after having computed the stratification.

- The algorithm tries a first cube partition : C itself, its faces, edges and corners.

- It starts with testing whether C is a Milnor ball.
  - It tries to find a point on the stratum in C that has the lowest dimension.
  - It tries σ^0, and finds no point in C (actually no point at all).
  - It tries σ^1, and finds the point $p := (0, 0, 0)$ inside C.
  - To test if C has conic structure, it tests the transversality for every pair of strata in $\mathcal{S}_C \times \mathcal{S}_Z$. We only detail the computation for the most interesting pair of strata.
    * It chooses the stratum $\sigma_C := [-1, 1] \times [-1, 1] \times \{1\} \in \mathcal{S}_C$.
      · It chooses σ^1 since σ^0 is empty and then σ^2. There is no critical point for the projection along the directions of σ_C.

      Consequently, $\sigma_C$is transverse to the strata of Z.
    * It chooses the 0-dimensional stratum $e := \sigma_C := \{1\} \times \{1\} \times \{1\}$ in $\mathcal{S}_C$.
      · It chooses σ^1 (σ_0 is empty) and check that there is no problem.
      · Next, it chooses σ^2, which intersects the interior of the segment $[p, e]$ at $(\frac{5}{3}, \frac{5}{3}, \frac{5}{3})$ hence the transversality test fails for σ_C.
- So C is not a Milnor ball.
- So the algorithm tries to refine the cube partition by splitting C in 4 identical cubes.

The algorithm goes on, until σ^2 does not intersect any more the segments joining the center to the vertices of the cubes, When this happens, graphically, the cubes are small enough so as to “follow” the surface. At this stage, the cube partition for the faces, edges and corners of the cube is already a conic cube partition.

The algorithm ends by putting the points on the 0-dimensional cells of the cube partition, then links these points to the points in the 1-dimensional cells, and so on. Finally we get the mesh of the surface that is shown above.

6 Conclusion

We stress the fact that the solutions we have given to the two problems “find a Whitney stratification” and “avoid tangency problems” is limited to three space.

As a matter of fact, one sees that theoretically there is a real gap between dimension 3 and higher dimensions as Speder theorem only guarantees the compatibility of two strata with codimension 1. Hence this way to compute a Whitney stratification is limited in nature to three space.

Algorithmically, the tests we do to avoid tangency problems would not generalize efficiently to higher dimensions as testing tangency through the projections would lead to the computation of iterated resultants. This would be doubly exponential in the number of variables as is the CAD algorithm. But the natural complexity of this problem is simply exponential in the number of variables.

Another issue which is not detailed here is about the complexity of this method and the number of boxes needed to approximate the (hyper)-surface within a given precision. Similar techniques as those described in [**?**], have to be investigated.

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